On the integrability of dynamical models with quadratic right-hand side

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Integrability of ODEs

Let us see the autonomous ODE system

$$\frac{d\,x_i}{d\,t} = \phi_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n.$$

Perhaps it has m independent functions of system variables $I_k(x_1, \ldots, x_n)$ such that complete differentiation with respect to the independent variable t is equal to zero along trajectories in the phase space of the system. We will talk about these functions as the first integrals of this system

$$\left.\frac{d\,I_k(x_1, \ldots, x_n)}{d\,t}\right|_{\frac{d\,x_i}{d\,t} = \phi_i(x_1, \ldots, x_n)} = 0, \quad k = 1, \ldots, m.$$

- The system can have m first integrals. We will say the system is integrable if it has enough of such (real) integrals.
- For integrability of an autonomous two-dimensional system, it is enough to have a single integral.

Simple Example

- Let us see the equation of the harmonic oscillator

$$\ddot{x}(t) + \omega_0^2 x(t) = 0$$

- This equation is equivalent to the system

$$\begin{cases} \dot{x}(t) = & y(t), \\ \dot{y}(t) = & -\omega^2 x(t). \end{cases} \qquad (1)$$

- The first integral of this system is

$$I(x(t), y(t)) = x^2(t) + y^2(t)/\omega_0^2.$$

- Due to (1) its full derivation in time is zero

$$\frac{d\, I(x(t), y(t))}{d\, t} = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)/\omega_0^2 = \\ = 2x(t)y(t) - 2x(t)y(t) = 0.$$

Solutions

Due to the constancy of the first integral

$$I(x(t), y(t)) = x^2(t) + y^2(t)/\omega_0^2 = C_1^2,$$

we evaluate $y(t)$ as a function of $x(t)$

$$y(t) = \pm\omega_0^2\sqrt{C_1^2 - x^2(t)}.$$

By substituting $y(t)$ in system (1) we get the autonomous equation of the first order

$$\frac{dx(t)}{dt} = \pm\omega_0^2\sqrt{C_1^2 - x^2(t)},$$

or

$$\pm\frac{dx(t)}{\omega_0^2\sqrt{C_1 - x^2(t)}} = dt, \quad \text{i.e. } x(t) = \pm C_1 \cdot \sin(\omega_0 t + C_2).$$

- Integrability is an important property of the system. In particular, if a system is integrable then it is solvable by quadrature.
- Knowledge of first integrals is also important for studying the phase portrait, bifurcation analysis, constructing symplectic integration schemes, etc.

Problem

- Generally, integrability is a rare property.
- But the system may depend on parameters.
- Our task here is to find the values of system parameters at which the system is integrable.

Kamke's Book

ГЛАВА IX

СИСТЕМЫ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

1—17. Системы двух дифференциальных уравнений

9.1 $x' = -x(x+y), \; y' = y(x+y)$.
Из этих уравнений получаем $yx' + xy' = 0$, т. е. $xy = C$. Таким образом, эта система сводится к одному уравнению с разделяющимися переменными $x' + x^2 + C = 0$.

9.2 $x' = (ay+b)x, \; y' = (cx+d)y$.
Из этих уравнений следует

$$\left(a + by^{-1}\right) y' = \left(c + dx^{-1}\right) x', \quad \text{т. е.} \quad y^b e^{ay} = Cx^d e^{cx}.$$

О дальнейшем исследовании решений в связи с некоторыми биологическими проблемами см. V. Volterra, *Rendiconti Sem. Mat. Milano* **3** (1930), стр. 158 и сл.

9.3 $x' = [a(px+qy)+\alpha]\,x, \; y' = [b(px+qy)+\beta]\,y$.
Отсюда следует $y^a x^{-b} = Ce^{(a\beta - b\alpha)t}$. См. V. Volterra, *Rendiconti Sem. Mat. Milano* **3** (1930), стр. 158 и сл. О дальнейшем исследовании этих уравнений в связи с некоторыми биологическими проблемами см. также A. J. Lotka, *Journ. Washington Acad.* **22** (1932), стр. 461—469; V. A. Kostitzin, *Actualités scientif.* **96** (1934).

9.4 $x' = h(a-x)(c-x-y), \; y' = k(b-y)(c-x-y)$.
Из этих уравнений следует $|y-b|^h = C|x-a|^k$. Таким образом, эта система может быть сведена к одному уравнению относительно x или y. Если в области $0 \leqslant x < a$, $0 \leqslant y < b$, $x+y < c$ требуется найти решение, удовлетворяющее начальным условиям $x(0) = y(0) = 0$, то получаем уравнение

$$x' = h(c-a-b)(a-x) + h(a-x)^2 + hba^{-k/h}(a-x)^{(k+h)/h}$$

и соответствующее уравнение для y. См. H. J. Curnow, *Journ. London Math. Soc.* **3** (1928), стр. 88—92. Подробное изучение этих уравнений в связи с некоторыми химическими проблемами см. J. G. van der Corput, H. J. Backer, *Proceedings Amsterdam* **41** (1938), стр. 1058—1073.

9.5 $x' = y^2 - \cos x, \; y' = -y \sin x$.
Отсюда следует $3y\cos x = y^3 + C$.
См. Е. Иконников, *ЖТФ* **4** (1937), стр. 433—437.

9.6. $x' = -xy^2 + x + y, \; y' = x^2 y - x - y$.
Решения удовлетворяют уравнению $x^2 + y^2 - 2\ln|xy-1| = C$.

Local Analysis and Resonance Normal Form

- The resonance normal form was introduced by H. Poincaré for the local investigation of systems of nonlinear ordinary differential equations. It is based on the maximal simplification of the right-hand sides of these equations by invertible transformations.
- The normal form approach was developed in works of G.D. Birkhoff, T.M. Cherry, A. Deprit, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno and others. This technique is based on the Local Analysis method by Prof. Bruno [Bruno 1971, 1972, 1979, 1989].

The Simplest Form of a Polynomial System

$$\begin{cases} \dot{x} = a + \alpha x + \beta y + P(x, y), \\ \dot{y} = b + \gamma x + \delta y + Q(x, y). \end{cases}$$

By shifting and similarity transformation

$$\begin{cases} \dot{\tilde{x}} = \lambda_1 \tilde{x} + \sigma \, \tilde{y} + \tilde{P}(\tilde{x}, \tilde{y}), \quad \sigma \neq 0 \quad \text{only if} \quad \lambda_1 = \lambda_2, \\ \dot{\tilde{y}} = \qquad \lambda_2 \tilde{y} + \tilde{Q}(\tilde{x}, \tilde{y}). \end{cases}$$

Ideally we wish to get a linear system

$$\begin{cases} \dot{\tilde{\tilde{x}}} = \lambda_1 \tilde{\tilde{x}} + \sigma \, \tilde{\tilde{y}}, \\ \dot{\tilde{\tilde{y}}} = \qquad \lambda_2 \, \tilde{\tilde{y}}. \end{cases}$$

This cannot be done throughout the entire phase space, but it is possible locally – at a small domain near the stationary point.

Power Series Approaches

- Newton. Solving differential equations in power series with respect to the independent variable near the stationary point.
- Poincare. Transformation of dependent variables in the form of power series in the neighborhood of a stationary point.

Local Integrability

We consider an autonomous system of ordinary differential equations

$$\frac{d\,x_i}{d\,t} \stackrel{\text{def}}{=} \dot{x}_i = \phi_i(X), \quad i = 1, \ldots, n, \tag{2}$$

where $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $\phi_i(X)$ are polynomials.

In a neighborhood of the point $X = X^0$, the system (2) is *locally integrable* if it has there sufficient number m of independent first integrals of the form

$$I_k(X) = \frac{a_k(X)}{b_k(X)}, \quad k = 1, \ldots, m,$$

where functions $a_k(X)$ and $b_k(X)$ are analytic in a neighborhood of this point. Such functions $I_k(X)$ are called the formal integral.

Multi-index Notation

Let's suppose that we treat the reduced to a diagonal polynomial system near a stationary point at the origin and rewrite this n-dimension system in the terms

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{\mathbf{q}\in\mathcal{N}_i} f_{i,\mathbf{q}} \mathbf{x}^{\mathbf{q}}, \quad i = 1, \ldots, n, \tag{3}$$

where we use the multi-index notation

$$\mathbf{x}^{\mathbf{q}} \equiv \prod_{j=1}^{n} {x_j}^{q_j}$$

with the power exponent vector $\mathbf{q} = (q_1, \ldots, q_n)$
Here the sets:

$$\mathcal{N}_i = \{\mathbf{q} \in \mathbb{Z}^n : q_i \geq -1 \text{ and } q_j \geq 0\,,\ \text{if } j \neq i\,, \quad j = 1, \ldots, n\}\,,$$

because the factor x_i has been moved out of the sum in (3).

Normal Form

The normalization is done with a near-identity transformation:

$$x_i = z_i + z_i \sum_{\mathbf{q}\in\mathcal{N}_i} h_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n \tag{4}$$

after which we have system (3) in the normal form:

$$\dot{z}_i = \lambda_i z_i + z_i \sum_{\substack{\langle \mathbf{q}, \mathbf{L}\rangle = 0 \\ \mathbf{q}\in\mathcal{N}_i}} g_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n, \tag{5}$$

where $\mathbf{L} = \{\lambda_1, \ldots, \lambda_n\}$ is the vector of eigenvalues.

Theorem (Bruno 1971)

There exists a formal change (4) reducing (3) to its normal form (5).

Note, the normalization (4) does not change the linear part of the system.

Resonance Terms

- The important difference between (3) and (5) is a restriction on the range of the summation, which is defined by the equation:

$$\langle \mathbf{q}, \mathbf{L} \rangle = \sum_{j=1}^{n} q_j \lambda_j = 0 . \qquad (6)$$

  I.e. the summation in the normal form (5) contains only terms, for which (6) is valid. They are called resonance terms.

  In the two-dimensional case there are a finite number of them, if the ratio of the eigenvalues is not non-positive rational.

- We rewrite below the normalized equation (5) as

$$\dot{z}_i = \lambda_i z_i + z_i g_i(Z), \qquad (7)$$

  where $g_i(Z)$ is the re-designate sum.

Calculation of the Normal Form

The h and g coefficients in (4) and (5) are found by using the recurrent formula:

$$g_{i,\mathbf{q}} + \langle \mathbf{q}, \mathbf{L} \rangle \cdot h_{i,\mathbf{q}} = -\sum_{j=1}^{n} \sum_{\substack{\mathbf{p}+\mathbf{r}=\mathbf{q} \\ \mathbf{p},\mathbf{r} \in \bigcup_i \mathcal{N}_i \\ \mathbf{q} \in \mathcal{N}_i}} (p_j + \delta_{ij}) \cdot h_{i,\mathbf{p}} \cdot g_{j,\mathbf{r}} + \tilde{\Phi}_{i,\mathbf{q}} \,, \quad (8)$$

For this calculation we have two programs.

- in LISP [Edneral, Khrustalev 1992]
- in the high-level language of the MATHEMATICA system [Edneral, Khanin 2002].

Conditions **A and** ω

There are two conditions

- Condition **A**. In the normal form (5)

$$g_j(Z) = \lambda_j a(Z) + \bar{\lambda}_j b(Z), \quad j = 1, \ldots, n, \tag{9}$$

  where $a(Z)$ and $b(Z)$ are some formal power series.
- Condition ω (on small divisors) [Bruno 1971]. It is fulfilled for almost all vectors **L**. At least it is satisfied at rational eigenvalues.

Theorem (Bruno 1971)

If vector **L** *satisfies Condition* ω *and the normal form* (5) *satisfies Condition* **A** *then the normalizing transformation (4) converges.*

Local Integral in the Resonance Case

Consider the case of a $[N : M]$ resonance in the two-dimension system. The eigenvalues here satisfy the ratio $N \cdot \lambda_1 = -M \cdot \lambda_2$ and from the condition **A** (9) we have

$$g_1(Z) = \lambda_1 a(Z) + \bar{\lambda}_1 b(Z), \quad g_2(Z) = \lambda_2 a(Z) + \bar{\lambda}_2 b(Z),$$

i.e. $N \cdot g_1(Z) = -M \cdot g_2(Z)$.
The normalized system (7) can be conditionally rewritten as

$$N \times \left| \frac{d\ \log(z_1)}{d\,t} = \lambda_1 + g_1(Z)\ , \quad -M \times \right| \frac{d\ log(z_2)}{d\,t} = \lambda_2 + g_2(Z)\ .$$

$$\text{So,} \quad \frac{d\ \log(z_1^N \cdot z_2^M)}{d\,t} = 0 \ \text{ or } \ z_1^N \cdot z_2^M = const.$$

It is the local first integral. So, the system is local integrable.

Near a stationary point the condition **A**:

- Ensures convergence;
- Provides the local integrability;

Problem

We will demonstrate the search for integrable cases using an example of the [Bautin 1952, Lunkevich Sibirskii 1982] system which has a quadratic polynomial right hand sides

$$\begin{aligned} \frac{d\tilde{x}(t)}{dt} &= \alpha\tilde{x}(t) + \beta\tilde{y}(t) + \tilde{\tilde{a}}_1\tilde{x}^2(t) + \tilde{\tilde{a}}_2\,\tilde{x}(t)\tilde{y}(t) + \tilde{\tilde{a}}_3\,\tilde{y}^2(t), \\ \frac{d\tilde{y}(t)}{dt} &= \gamma\tilde{x}(t) + \delta\tilde{y}(t) + \tilde{\tilde{b}}_1\tilde{x}^(t) + \tilde{\tilde{b}}_2\,\tilde{x}(t)\tilde{y}(t) + \tilde{\tilde{b}}_3\,\tilde{y}^2(t), \end{aligned} \tag{10}$$

where $\tilde{x}(t)$ and $\tilde{y}(t)$ are functions in time and other letters are arbitrary real parameters.

By a linear transformation, the linear part of this system can be reduced to the Jordan form. The origin is the stationary point now

$$\begin{aligned} \dot{x} &= \lambda_1\,x + \sigma\;\; y + \tilde{a}_1x^2 + \tilde{a}_2\,x\,y + \tilde{a}_3y^2, \\ \dot{y} &= \qquad\quad \lambda_2\,y + \tilde{b}_1x^2 + \tilde{b}_2\,x\,y + \tilde{b}_3y^2. \end{aligned}$$

We omit here the time dependence and use a dot instead of the time derivative. λ_1, λ_2 are eigenvalues. If $\lambda_1 \neq \lambda_2$ then $\sigma = 0$.

Two Cases

If both eigenvalues are non-zero at the same time, we can choose $|\lambda_2| = 1$ using time scaling. Also we put $\sigma = 0$. Then we have two resonance cases of system (10). The center case

$$\begin{aligned} \dot{x} &= \quad i\,y + a_1\,x^2 + a_2\,x\,y + a_3 y^2, \\ \dot{y} &= -i\,x + b_1\,x^2 + b_2\,x\,y + b_3 y^2, \end{aligned}$$

and the saddle case

$$\begin{aligned} \dot{x} &= \lambda\,x + a_1\,x^2 + a_2\,x\,y + a_3 y^2, \\ \dot{y} &= -y + b_1\,x^2 + b_2\,x\,y + b_3 y^2, \end{aligned}$$

where $0 \geq \lambda \in \mathbb{Z}$.

Phase Portrait near Stationary (Equilibrium) Points

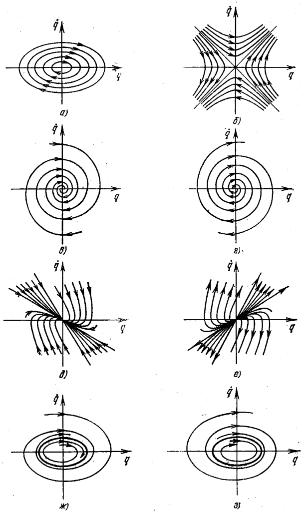


Рис. 17.17. Особые точки на фазовой плоскости; *а*) центр, *б*) седло; *в*) фокус (устойчивый). *г*) фокус (неустойчивый); *д*) узел (устойчивый); *е*) узел (неустойчивый); *ж*, *з*) изолированные циклы (устойчивый и неустойчивый). Об устойчивости и неустойчивости см. ниже.

Integrable Center Case

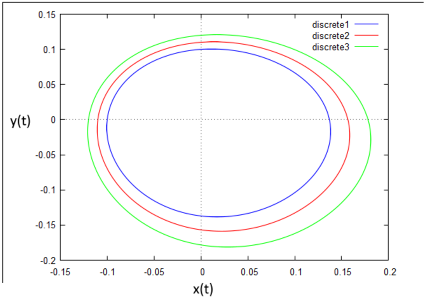
Bautin's System, Center-Focus Case:

x'(t) = y(t) + x(t)^2 + x(t)*y(t) + y(t)^2

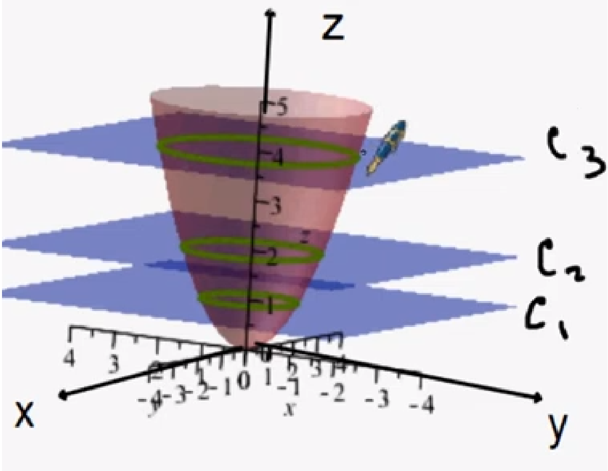
y'(t) = - x(t) + x(t)^2 + x(t)*y(t) + y(t)^2

$x(0) = 0$, $y(0) = 0.1, 0.11, 0.12$

Center:



The level line is closed near a local extremum of the real first integral



Non-integrable Focus Case

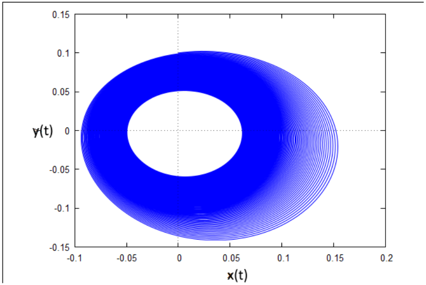
Bautin's System, Center-Focus Case:

x'(t) = y(t) + x(t)^2 + x(t)*y(t) + y(t)^2

y'(t) = - x(t) + x(t)^2 + x(t)*y(t) + 2*y(t)^2

x(0) = 0, y(0) = 0.1

Weak Focus



Integrable Saddle Case

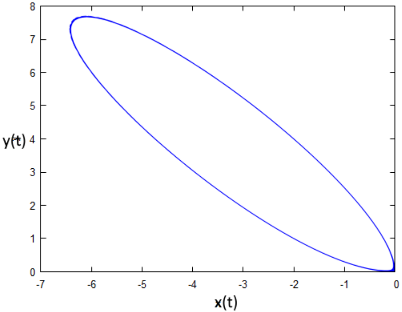
Bautin Saddle Integrable System (B6)

x'(t) = x(t) + x(t)*y(t) + y(t)^2

y'(t) = - y(t) + x(t)^2 + 1/2 *y(t)^2

x0 = -0.1, y0 = 0

2001



Non-integrable Saddle Case

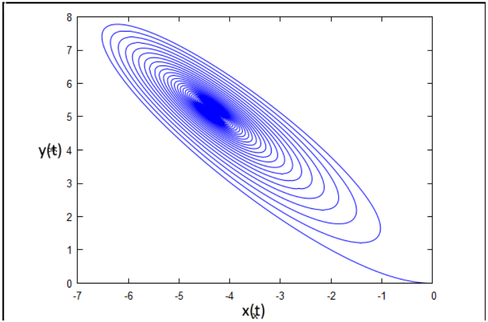
Bautin Saddle non-Integrable System (B6)

x'(t) = x(t) + x(t)*y(t) + y(t)^2

y'(t) = - y(t) + x(t)^2 + 1/2.01*y(t)^2

x(0) = -0.1, y(0) = 0

1:



Hypothesis

We seek integrability by solving the local integrability condition at all stationary points of the system with resonances in the linear parts.The basis of our technique can be formulated as a hypothesis

Hypothesis

For the existence of the first integral in a certain domain of the ODEs phase space, local integrability is required in the neighborhood of each stationary point in this domain.

At resonance cases the local integrability can be written down as the system of algebraic equations on the system parameters. We can create this system by the computer program.

Condition of Local Integrability as a System of Algebraic Equations

For the saddle case and the resonance 1 : 1 the truncated **A** system at the origin stationary point is

$$a_1 a_2 - b_2 b_3 = 0,$$

$$-a_3 b_2(-6a_1^2 + 9a_1 b_2 + 14 b_1 b_3 + 6b_2^2) + 9a_2^2(a_1 b_2 + b_1 b_3) + a_2(14 a_1 a_3 b_1 - 3b_3(2b_1 b_3 + 3b_2^2)) + 6a_2^3 b_1 = 0,$$

$$432 a_1^4 a_2 a_3 + 36 a_1^3(54 a_2^3 + 18 a_2^2 b_3 - 61 a_2 a_3 b_2 - 18 a_3 b_2 b_3) - 6a_1^2(162 a_2^3 b_2 + a_2^2(131 a_3 b_1 - 162 b_2 b_3) + 3a_2 a_3(106 b_1 b_3 + 75 b_2^2) + 2a_3 b_2(194 a_3 b_1 - 381 b_2 b_3)) + a_1(3708 a_2^4 b_1 - 108 a_2^3(33 b_2^2 - 38 b_1 b_3) - 3a_2^2 b_1(5299 a_3 b_2 + 1524 b_3^2) - 4a_2(868 a_3^2 b_1^2 - 981 a_3 b_2^3 + 81 b_3^2(3b_2^2 - 2b_1 b_3)) + 36 b_2(142 a_3^2 b_1 b_2 + a_3 b_3(53 b_1 b_3 - 114 b_2^2) - 18 b_2 b_3^3)) - 1782 a_2^4 b_1 b_2 - 6a_2^3 b_1(523 a_3 b_1 + 654 b_2 b_3) + 18 a_2^2 b_3(-284 a_3 b_1^2 + 75 b_1 b_2 b_3 + 198 b_2^3) + 3a_2(a_3(776 b_1^2 b_3^2 + 5299 b_1 b_2^2 b_3 + 594 b_2^4) + 12 b_2 b_3^2(61 b_1 b_3 + 27 b_2^2)) + 2b_2(a_3^2 b_1(1736 b_1 b_3 + 1569 b_2^2) + 3a_3 b_2 b_3(131 b_1 b_3 - 618 b_2^2) - 108 b_3^3(2b_1 b_3 + 9b_2^2)) = 0.$$

It has been experimentally established that adding further equations does not change solutions of this system.

Equations of a similar form were obtained also for resonances 2 : 1 and 3 : 1 and for pure imaginary eigenvalues also.

Solutions of the Condition **A**

The MATHEMATICA-11 system solver Solve received 13 families of rational solutions of the algebraic system above. Some of them are a consequence of others, we marked them by asterisks:

1) $\{a_1 = \frac{b_2 b_3}{a_2}, b_1 = \frac{a_3 b_2^3}{a_2^3}\};$

2) $\{a_2 = 0, b_2 = 0\};$

3) $\{a_1 = -\frac{b_2}{2}, b_3 = -\frac{a_2}{2}\};$

4) $^*\{a_1 = -\frac{b_2}{2}, a_2 = 0, b_3 = 0\};$

5) $\{a_2 = 0, a_3 = 0, b_3 = 0\};$

6) $^*\{a_1 = 0, b_2 = 0, b_3 = -\frac{a_2}{2}\};$

7) $\{a_1 = 2b_2, b_1 = \frac{a_2 b_2}{a_3}, b_3 = 2a_2\};$

8) $^*\{a_1 = 2b_2, b_1 = \frac{a_3 b_2^3}{a_2^3}, b_3 = 2a_2\};$

9) $^*\{a_1 = 2b_2, a_2 = 0, a_3 = 0, b_3 = 0\};$

10) $^*\{a_1 = 2b_2, a_2 = 0, b_1 = 0, b_3 = 0\};$

11) $^*\{a_1 = 0, a_3 = 0, b_2 = 0, b_3 = 2a_2\};$

12) $\{a_1 = -\frac{b_2}{2}, a_3 = 0, b_1 = 0, b_3 = -\frac{a_2}{2}\};$

13) $\{a_1 = 2b_2, a_3 = 0, b_1 = 0, b_3 = 2a_2\}.$

With these sets of parameters, we checked, if possible, the integrability condition at other stationary points of the system.

Calculation of the First Integrals

An autonomous second order system can be rewritten as a non-autonomous first order equation. Let

$$\frac{d\,x(t)}{d\,t} = P(x(t), y(t)), \quad \frac{d\,y(t)}{d\,t} = Q(x(t), y(t)).$$

We divided the left and right hand sides of the system equations into each other. In result we have the first-order non-autonomous differential equation for $x(y)$ or $y(x)$

$$\frac{d\,x(y)}{d\,y} = \frac{P(x(y), y)}{Q(x(y), y)} \quad \text{or} \quad \frac{d\,y(x)}{d\,x} = \frac{Q(x, y(x))}{P(x, y(x))}.$$

Then we try to solve them by the MATHEMATICA-11 solver DSolve and got the solution $y(x)$ (or $x(y)$). After that we calculated the integral from this solution by extracting the integration constant.

If this procedure failed, we manually used the Darboux method.

First Integrals

We calculated the integrals for the resonance 1:1 case:

1) $\dot{x} = x + b_2 b_3 x^2/a_2 + a_2 xy + a_3 y^2, \quad \dot{y} = -y + a_3 b_2^3 x^2/a_2^3 + b_2 xy + b_3 y^2,$

$$I_1(x,y) = \left((a_3 b_2 - a_2 b_3)(b_2 x - a_2 y) - a_2^2\right) \times \left(b_2^2 \left(1 - \frac{(a_2 b_3 - a_3 b_2)(a_2 y - b_2 x)}{a_2^2}\right)\right)^{\frac{a_2(a_2+2b_3)}{a_3 b_2 - a_2 b_3}} \times \left(2a_2^4 a_3 + (a_2(a_2+b_3) + a_3 b_2)\left(a_3 b_2^2 x^2 (a_2^2 + 2a_3 b_2) + a_2 x \left(y(a_2^2 + 2a_3 b_2)(a_2^2 - a_2 b_3 + a_3 b_2) + 2a_2 a_3 b_2\right) + a_2^2 a_3 y(y\left(a_2^2 + 2a_3 b_2\right) - 2a_2)\right)\right);$$

2) $\dot{x} = x + a_1 x^2 + a_3 y^2, \quad \dot{y} = -y + b_1 x^2 + b_3 y^2;$

$$I_1(x,y) = b_1(a_3 b_1 - a_1 b_3) \int (-y + b_1 x^2 + b_3 y^2) \times \left(x\left(-x\left(a_1^2 + b_1 x(a_1 b_3 - a_3 b_1) + b_1 b_3\right) - 2a_1\right) - y^2\left(b_3 x(a_1 b_3 - a_3 b_1) + a_1 a_3 + b_3^2\right) + y(2b_3 - x(a_1 x + 3)(a_3 b_1 - a_1 b_3)) + a_3 y^3 (a_1 b_3 - a_3 b_1) - 1\right)^{-1} dx;$$

3) $\dot{x} = x - \frac{1}{2} b_2 x^2 + a_2 xy + a_3 y^2, \quad \dot{y} = -y + b_1 x^2 + b_2 xy - \frac{1}{2} a_2 y^2,$

$I_3(x, y) = -3a_2 xy^2 - 2a_3 y^3 + 2b_1 x^3 + 3b_2 x^2 y - 6xy;$

4) $\dot{x} = x - \frac{1}{2} b_2 x^2 + a_3 y^2, \quad \dot{y} = -y + b_1 x^2 + b_2 xy,$

$I_4 = -\frac{2}{3} a_3 y^3 + \frac{2}{3} b_1 x^3 + xy(b_2 x - 2);$

5) $\dot{x} = x + a_1 x^2, \quad \dot{y} = -y + b_1 x^2 + b_2 xy,$

$$I_5(x, y) = \frac{(a_1 x+1)^{-\frac{b_2}{a_1}-1}}{b_2(a_1-b_2)(a_1+b_2)} \left(a_1^2 b_2 xy - a_1 b_1 b_2 x^2 - 2a_1 b_1 x - b_1 b_2^2 x^2 - 2b_1 b_2 x - 2b_1 + b_2^3(-x)y\right);$$

6) $\dot{x} = x + a_2 xy + a_3 y^2, \quad \dot{y} = -y + b_1 x^2 - \frac{1}{2} a_2 y^2,$

$I_6 = xy(a_2 y + 2) + \frac{2}{3} a_3 y^3 - \frac{2}{3} b_1 x^3;$

7) $\dot{x} = x + 2b_2 x^2 + a_2 xy + a_3 y^2, \quad \dot{y} = -y + \frac{a_2 b_2}{a_3} x^2 + b_2 xy + 2a_2 y^2,$

$I_7(x, y) = \frac{a_2 b_2 x^2}{a_3} + 2a_2 y^2 + b_2 xy - y;$

8) $\dot{x} = x + 2b_2x^2 + a_2xy + a_3y^2, \quad \dot{y} = -y + \frac{a_3b_2^3}{a_2^3}x^2 + b_2xy + 2a_2y^2,$

$$I_8 = \left(\left(a_3b_2 - 2a_2^2\right)(b_2x - a_2y) - a_2^2\right)\left(b_2^2\left(1 - \frac{\left(2a_2^2 - a_3b_2\right)(a_2y - b_2x)}{a_2^2}\right)\right)^{\frac{5a_2^2}{a_3b_2 - 2a_2^2}} \times$$
$$\left(2a_2^4a_3 + \left(3a_2^2 + a_3b_2\right)\left(a_3b_2^2x^2\left(a_2^2 + 2a_3b_2\right) + \right.\right.$$
$$\left.\left. a_2x\left(y\left(a_3b_2 - a_2^2\right)\left(a_2^2 + 2a_3b_2\right) + 2a_2a_3b_2\right) + a_2^2a_3y\left(y\left(a_2^2 + 2a_3b_2\right) - 2a_2\right)\right)\right)$$

9) $\dot{x} = x + 2b_2x^2, \quad \dot{y} = -y + b_1x^2 + b_2xy,$

$I_9 = (3b_2^3xy - b_1(3b_2x(b_2x + 2) + 2))/(3b_2^3(2b_2x + 1)^{3/2});$

10) $\dot{x} = x + 2b_2x^2 + a_3y^2, \quad \dot{y} = -y + b_2xy,$

$I_{10}(x, y) = \frac{a_3b_2^2\left(\frac{1}{3}\log\left(a_3y^2 + 3x\right) - \frac{1}{2}\log\left(a_3b_2y^2 + 2b_2x + 1\right)\right)}{b_2 + 1} + \frac{a_3b_2^2\log(3b_2y + 3y)}{3(b_2 + 1)};$

11) $\dot{x} = x + a_2xy, \quad \dot{y} = -y + b_1x^2 + 2a_2y^2,$

$I_{11}(x, y) = \frac{a_2^2b_1\log(x)}{3(a_2 - 1)} - \frac{a_2^2b_1\left(\frac{1}{2}\log\left(a_2b_1x^2 - 2a_2y + 1\right) - \frac{1}{3}\log\left(3y - b_1x^2\right)\right)}{a_2 - 1};$

12) $\dot{x} = x - b_2/2x^2 + a_2xy, \quad \dot{y} = -y + b_2xy - a_2/2y^2,$

$I_{12}(x, y) = \frac{a_2xy^2 - b_2x^2y + 2xy}{a_2};$

13) $\dot{x} = x + 2b_2x^2 + a_2xy, \quad \dot{y} = -y + b_2xy + 2a_2y^2,$

$$I_{13}(x, y) = (216a_2^3y^3 - 648a_2^2b_2xy^2 - 324a_2^2y^2 + 648a_2b_2^2x^2y + 648a_2b_2xy + 162a_2y - 216b_2^3x^3 - 324b_2^2x^2 - 162b_2x - 27)/(x^2y^2).$$

Nonintegrable Case?

We have carried out similar calculations for the case of pure imaginary eigenvalues and got 20 appropriate sets of parameters (11 independent). We found integrability for all sets.

Also we did that for the resonance 2:1 and got 12 sets of parameters (8 independent). 7 of them correspond to integrable cases. But for one

$$\dot{x} = 2x - \frac{1}{2} b_3 xy, \quad \dot{y} = -y + b_1 x^2 + b_3 y^2,$$

we could not find the first integral. This case needs further research.

Non-resonant Case

But the algebraic systems for each resonance have a similar form and are written with respect to the same variables. That's why the next step was to combine algebraic equations for local integrability conditions for all three calculated resonances 1 : 1, 2 : 1 and 3 : 1. In other words, we tried to extrapolate the results of resonant cases to the situation with arbitrary eigenvalues. Of course, such extrapolation does not describe all integrable cases.

The solutions of the resulting system of this 9 equations can predict the integrable cases of the general system

$$\begin{aligned} \dot{x} &= \alpha\, x + a_1\, x^2 + a_2\, x\, y + a_3\, y^2, \\ \dot{y} &= -\, y + b_1\, x^2 + b_2\, x\, y + b_3\, y^2, \end{aligned} \tag{11}$$

where α is an arbitrary parameter.

Solutions of the Combined System

The system has 14 rational solutions of the system above. Some of them are a consequence of others. 11 solutions are independent:

1) $\{a_2 = 0, a_3 = 0, b_2 = 0\}$;
2) $\{a_2 = 0, a_3 = 0, b_3 = 0\}$;
3) $\{a_1 = 0, b_1 = 0, b_2 = 0\}$;
4) $^*\{a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0\}$;
5) $\{a_1 = 2b_2, a_2 = 0, b_1 = 0, b_3 = 0\}$;
6) $\{a_1 = 0, a_3 = 0, b_1 = 0, b_3 = 0\}$;
7) $\{a_1 = 0, b_1 = 0, b_2 = 0, b_3 = 0\}$;
8) $\{a_1 = 0, b_1 = 0, b_2 = 0, b_3 = -\frac{a_2}{2}\}$;
9) $\{a_1 = b_2, a_3 = 0, b_1 = 0, b_3 = a_2\}$;
10) $\{a_1 = 0, b_1 = 0, b_2 = 0, b_3 = a_2\}$;
11) $\{a_1 = 0, a_3 = 0, b_2 = 0, b_3 = 2a_2\}$;
12) $\{a_1 = 0, b_1 = 0, b_2 = 0, b_3 = 2a_2\}$;
13) $^*\{a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0, b_3 = 0\}$;
14) $^*\{a_1 = 0, a_3 = 0, b_1 = 0, b_2 = 0, b_3 = -\frac{a_2}{2}\}$.

For all sets of parameters above we found the first integrals.

Integrals of the Non-resonant System

1) $\dot{x} = \alpha x + a_1 x^2, \quad \dot{y} = -y + b_1 x^2 + b_3 y^2,$

This is the integrable case, but the expression for the first integral is too huge for a demonstration here.

2) $\dot{x} = \alpha x + a_1 x^2, \quad \dot{y} = -y + b_1 x^2 + b_2 xy,$

$$I_2(x,y) = \frac{x^{1/\alpha}(\alpha + a_1 x)^{-\frac{1}{\alpha} - \frac{b_2}{a_1}}}{(\alpha+1)a_1} \times \Bigg(\alpha b_1 x \left(\frac{a_1 x}{\alpha} + 1\right)^{\frac{1}{\alpha} + \frac{b_2}{a_1}} {}_2F_1\left(1 + \frac{1}{\alpha}, \frac{b_2}{a_1} + \frac{1}{\alpha}; 2 + \frac{1}{\alpha}; -\frac{a_1 x}{\alpha}\right) - \alpha b_1 x \left(\frac{a_1 x}{\alpha} + 1\right)^{\frac{1}{\alpha} + \frac{b_2}{a_1}} {}_2F_1\left(1 + \frac{1}{\alpha}, \frac{b_2}{a_1} + \frac{1}{\alpha} + 1; 2 + \frac{1}{\alpha}; -\frac{a_1 x}{\alpha}\right) - \alpha a_1 y - a_1 y \Bigg);$$

3) $\dot{x} = \alpha x + a_2 xy + a_3 y^2, \quad \dot{y} = -y + b_3 y^2,$

$$I_3(x,y) = \frac{y^{\alpha}(1 - b_3 y)^{-\alpha - \frac{a_2}{b_3}}}{\alpha + 2} \times \left(a_3 y^2 (1 - b_3 y)^{\alpha + \frac{a_2}{b_3}} {}_2F_1\left(\alpha + 2, \frac{a_2 + b_3 + b_3\alpha}{b_3}; \alpha + 3; b_3 y\right) + \alpha x + 2x \right);$$

4) $\dot{x} = \alpha x + a_3 y^2, \quad \dot{y} = -y + b_3 y^2,$

$$I_4(x,y) = \frac{e^{-\alpha(\log(1 - b_3 y) - \log(y))}}{(\alpha+1)b_3} \times \Big(a_3 y^{\alpha+1} {}_2F_1(\alpha, \alpha+1; \alpha+2; b_3 y) e^{\alpha(\log(1-b_3 y) - \log(y))} - a_3 y^{\alpha+1} {}_2F_1(\alpha+1, \alpha+1; \alpha+2; b_3 y) e^{\alpha(\log(1-b_3 y) - \log(y))} - \alpha b_3 x - b_3 x \Big)$$

5) $\dot{x} = \alpha x + 2b_2x^2 + a_3y^2, \quad \dot{y} = -y + b_2xy,$

$$I_5(x, y) = \frac{a_3 b_2^2}{\alpha(\alpha+2)(b_2+1)} \times \Big(-\alpha \log\left(\alpha + a_3b_2y^2 + 2b_2x\right) - 2\log\left(\alpha + a_3b_2y^2 + 2b_2x\right) + 2\log\left(a_3y^2 + (\alpha+2)x\right) + 2\alpha\log(y)\Big)$$

6) $\dot{x} = \alpha x + a_2xy, \quad \dot{y} = -y + b_2xy,$

$I_6(x, y) = -b_2x + a_2y + \log(x) + \alpha\log(y);$

7) $\dot{x} = \alpha x + a_2xy + a_3y^2, \quad \dot{y} = -y,$

$I_7(x, y) = y^\alpha(-a_2y)^{-\alpha}\left(a_2^2xe^{a_2y}(-a_2y)^\alpha - a_3\Gamma(\alpha+2, -a_2y)\right)/a_2^2;$

8) $\dot{x} = \alpha x + a_2xy + a_3y^2, \quad \dot{y} = -y - \frac{1}{2}a_2y^2,$

$$I_8(x, y) = \frac{y^\alpha}{(\alpha+2)(\alpha+3)(a_2y+2)^\alpha} \times \Big(2a_3y^2\left(\tfrac{1}{2}a_2y + 1\right)^\alpha \Big(2(\alpha+3)\,{}_2F_1\left(\alpha, \alpha+2; \alpha+3; -\tfrac{1}{2}a_2y\right) + (\alpha+2)a_2y\,{}_2F_1\left(\alpha, \alpha+3; \alpha+4; -\tfrac{1}{2}a_2y\right)\Big) + (\alpha+2)(\alpha+3)x(a_2y+2)^2\Big);$$

9) $\dot{x} = \alpha x + b_2x^2 + a_2xy, \quad \dot{y} = -y + b_2xy + a_2y^2,$

$I_9(x, y) = \frac{xy^\alpha}{b_2}(\alpha - \alpha\, a_2y + b_2x)^{-\alpha-1};$

10) $\dot{x} = \alpha x + a_2 xy + a_3 y^2, \quad \dot{y} = -y + a_2 y^2,$

$$I_{10}(x,y) = \frac{y^{\alpha}}{(\alpha+1)a_2(a_2y-1)(1-a_2y)^{\alpha}} \times \Big(a_2 a_3 y^2 (1-a_2y)^{\alpha}\, {}_2F_1(\alpha+1,\alpha+1;\alpha+2;a_2y) - a_3 y(1-a_2y)^{\alpha}\, {}_2F_1(\alpha+1,\alpha+1;\alpha+2;a_2y) + \alpha a_2 x + a_2 x + a_3 y\Big);$$

11) $\dot{x} = \alpha x + a_2 xy, \quad \dot{y} = -y + b_1 x^2 + 2a_2 y^2,$

$$I_{11}(x,y) = \frac{a_2^2 b_1 x^2 \left(-b_1x^2+2\alpha y+y\right)^{2\alpha}}{\alpha(2\alpha+1)(a_2-\alpha)\left(\alpha(2a_2y-1)-a_2b_1x^2\right)^{2\alpha+1}};$$

12) $\dot{x} = \alpha x + a_2 xy + a_3 y^2, \quad \dot{y} = -y + 2a_2 y^2,$

$$I_{12}(x,y) = \frac{y^{\alpha}(1-2a_2y)^{-\alpha-\frac{1}{2}}}{\alpha+2} \times \left(a_3 y^2 (1-2a_2y)^{\alpha+\frac{1}{2}}\, {}_2F_1\left(\alpha+\tfrac{3}{2},\alpha+2;\alpha+3;2a_2y\right) + \alpha x + 2x\right);$$

13) $\dot{x} = \alpha x + a_3 y^2, \quad \dot{y} = -y,$

$I_{13}(x,y) = y^{\alpha}(2x + \alpha x + a_3 y^2)/(2+\alpha);$

14) $\dot{x} = \alpha x + a_2 xy, \quad \dot{y} = -y - \frac{1}{2} a_2 y^2,$

$I_{14}(x,y) = xy^{\alpha}(a_2 y + 2)^{2-\alpha}.$

Examples 9.1 – 9.4 and 9.6 of chapter IX of the book [Kamke] are examples of integrable cases of systems of two autonomous ODEs with quadratic polynomial right-hand sides. Systems 9.1 and 9.6 have the linear parts with all zero eigenvalues and are outside the scope of this discussion. Other examples are:

- System 9.2 $\quad \dot{x} = x(ay + b), \quad \dot{y} = y(cx + d)$, after changing the time $t \to -\tau/d$ goes to case 6 above, if we substitute $\alpha \to -b/d, a_2 \to -a/d, b_2 \to -c/d$;
- System 9.3 $\quad \dot{x} = x\,[a(px + qy) + \alpha], \quad \dot{y} = y\,[b(px + q) + \beta]$. Case 9 above is its special case at $a = b$ by changing the time and parameters α, a_2 and b_2 ;
- System 9.4 $\quad \dot{x} = h(a - x)(c - x - y), \quad \dot{y} = k(b - y)(c - x - y)$, by the shift $x \to x + a, y \to y + b$ is reduced to the form with a stationary point at the origin $\dot{x} = h\,x\,(a + b - c + x + y), \quad \dot{y} = k\,y\,(a + b - c + x + y)$, and also can be transformed to case 9 at the special case $h = k$.

So, our results are consistent with this book.

Other Examples

We treated the degenerated system [Bruno, Edneral, Romanovski 2017], [Bruno, Edneral 2024]

$$\begin{cases} \dot{x} = -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2 \\ \dot{y} = (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3 \end{cases}$$

with five arbitrary real parameters $b \neq 0, a_1, a_2, b_1, b_2$.

With this technique, we found 7 parameter sets at which the system above is integrable.

For the Liénard-type system

$$\begin{cases} \dot{x} = y \\ \dot{y} = (a_0 + a_1 x)\, y + b_1 x + b_2 x^2 + b_3 x^3 \end{cases}$$

with five arbitrary real parameters a_0, a_1, b_1, b_2, b_3 we we found 6 parameter sets at which the system is integrable [Edneral 2023].

So we have results for more than just the homogeneous quadratic right-hand side.

Integrable Cases of a Three-dimensional Problem

First we considered resonant cases of the system

$$\begin{aligned} \dot{x} &= \alpha\, x + a_2\, x\, y + a_4\, x\, z + a_5\, y\, z, \\ \dot{y} &= -\beta\, y + b_2\, x\, y + b_4\, x\, z + b_5\, y\, z, \\ \dot{z} &= -\, z + c_2\, x\, y + c_4\, x\, z + c_5\, y\, z \end{aligned} \tag{12}$$

with natural α, β on the square table $\{1,2,3\} \times \{1,2,3\}$

N	α	β	Algebraic solutions	ODEs Solutions	% Success
8	1	1	23	19	83
8	1	2	16	12	75
8	1	3	25	19	76
8	2	1	57	49	86
8	2	2	34	29	85
8	2	3	43	35	81
9	3	1	60	51	85
9	3	2	63	58	92
10	3	3	43	38	88
Σ			364	310	

Then we solved the combined algebraic system of 329 equations, found its 10 solutions, and opened that MATHEMATICA-11 system solved all corresponding systems of ODEs of the form (12) except one (a red color). These systems with arbitary α and β are:

$$
\begin{array}{lll}
\dot{x} = \alpha x + a_2 x\,y + a_4 x\,z + a_5 y\,z, & \dot{y} = -\beta y + b_5 y\,z, & \dot{z} = -z + c_5 y\,z; \\
\dot{x} = \alpha x, & \dot{y} = -\beta y + b_2 x\,y + b_4 x\,z, & \dot{z} = -z + c_4 x\,z; \\
\dot{x} = \alpha x + a_2 x\,y + a_4 x\,z + a_5 y\,z, & \dot{y} = -\beta y + a_4 y\,z, & \dot{z} = -z - a_2 y\,z; \\
\dot{x} = \alpha x, & \dot{y} = -\beta y + b_2 x\,y, & \dot{z} = -z + c_4 x\,z; \\
\dot{x} = \alpha x, & \dot{y} = -\beta y + b_4 x\,z, & \dot{z} = -z + c_4 x\,z; \\
\dot{x} = \alpha x, & \dot{y} = -\beta y, & \dot{z} = -z + c_4 x\,z + c_5 y\,z; \\
\dot{x} = \alpha x, & \dot{y} = -\beta y + b_2 x\,y + b_5 y\,z, & \dot{z} = -z; \\
\dot{x} = \alpha x + a_4 x\,z, & \dot{y} = -\beta y + b_4 x\,z + a_4 y\,z, & \dot{z} = -z; \\
\dot{x} = \alpha x + a_5 y\,z, & \dot{y} = -\beta y + b_2 x\,y, & \dot{z} = -z - b_2 x\,z; \\
\dot{x} = \alpha x, & \dot{y} = -\beta y, & \dot{z} = -z + c_4 x\,z.
\end{array}
$$

However, the Maple-17 system gives finite solutions for all cases above.

Of course, such extrapolation describes only the simplest cases.

Non-resonant Three-dimensional System

Finally, we considered the general case of a three-dimensional system with 20 parameters

$$\begin{aligned}
\dot{x} &= \alpha\, x + a_1\, x^2 + a_2\, x\, y + a_3\, y^2 + a_4\, x\, z + a_5\, y\, z + a_6\, z^2, \\
\dot{y} &= -\beta\, y + b_1\, x^2 + b_2\, x\, y + b_3\, y^2 + b_4\, x\, z + b_5\, y\, z + b_6\, z^2, \\
\dot{z} &= -z + c_1\, x^2 + c_2\, x\, y + c_3\, y^2 + c_4\, x\, z + c_5\, y\, z + c_6\, z^2.
\end{aligned}$$

Calculating the normal form up to 6th order for 4 integer pairs $\{\alpha, \beta\}$= $\{1, 1\}$, $\{1, 2\}$, $\{2, 1\}$ and $\{2, 2\}$, we got a system of 121 equations for 18 parameters. We found 174 solutions for it. For 109 of the found sets of parameters the MATHEMATICS-13.3.1.0 system calculated solutions to the corresponding dynamical systems.

3D Chemical Kinetics Models

- There are many cases of integrability of three-dimensional systems and the corresponding exact solutions can be useful in applications, for example, in problems of chemical kinetics [Levanov, Antipenko 2006].
- The explicit form of solutions allows us to study the qualitative picture of the phase space of the system depending on the parameters, and bifurcation analysis can allow us to discover new phenomena in the simulated processes.

Jabotinsky-Korzukhin model

Let's consider, for example, the Jabotinsky-Korzukhin model [Korzukhin, Jabotinsky 1965].

$$\begin{aligned} \dot{x} &= k_1 x(C - y) - k_0 x\, z, \\ \dot{y} &= k_1 x(C - y) - k_2 y, \\ \dot{z} &= k_2 y - k_3 z. \end{aligned}$$

The eigenvalues of the linear part of this system are equal to

$$\{C \cdot k_1, -k_2, -k_3\}.$$

- After diagonalization, the linear part of the Jabotinsky-Korzukhin model takes the form with the quadratic right hand side.
- The question arises: under what additional conditions does the diagonalized version of the Jabotinsky-Korzukhin system appear among the exactly solvable cases of the integrable system?
- We found that the model system has 5 cases integrability by quadratures if the following relations on its parameters are satisfied

$$k_0 = \frac{C \cdot k_1 + 1}{C}, \quad k_2 = -C \cdot k_1, \quad k_3 = 1.$$

- Unfortunately, the coefficients in the real model must be positive, so the requirement above is not feasible in reality. But this example illustrates the possibility of finding exactly solvable cases in dynamic models.

Note, in many interesting cases the linear parts of ODEs of the dynamical models can be reduced to Jordan form, but not to diagonal form.

At present, the author is not aware of computer packages for normalizing ODE systems with non-diagonalizable linear parts, but appropriate formulas exist, see [Bruno 1979].

The author hopes to close this gap in the future. Then it will be possible to significantly expand the range of search for integrable cases of multidimensional systems.

Conclusions

- There is a empirical technique for searching for analytically solvable cases of dynamical systems. This works both for the case of resonance in the linear part of the system, and for the general case.
- There are many cases of integrability of multidimensional dynamic systems. The corresponding exact solutions can be useful in applications, for example, in bifurcation analysis of various models.
- To study the integrability of such systems, it is necessary to create packages for reducing ODE systems with a Jordan matrix of the linear part to the normal form.

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