On the integrability of dynamical models with quadratic right-hand side

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Abstract. We use a heuristic method that allows us to determine in advance the cases of integrability of autonomous dynamical systems with a polynomial right-hand side. The capabilities of the method are demonstrated using examples of two- and three-dimensional systems with quadratic nonlinearity on the right side. Application of the discussed approach allows us to find many integrable cases of such systems, which can be used in the study of mathematical models.

Introduction

In previous works [1, 2] a technique was described for constructing some systems of algebraic equations for the parameters of an ODE system with resonance in the linear part. It was experimentally shown that using relations on the parameters obtained as a result of solving such systems, one can find explicit expressions for the first integrals or solutions of ODEs in quadratures. It was also discovered that by considering integrability conditions simultaneously for several resonances, it is possible to obtain integrability conditions for general (non-resonant) cases.

The talk discusses the use of this method for finding first integrals of two- and three-dimensional systems with quadratic nonlinearity and possible applications of these results in modeling.

1. Two-dimensional case

First we considered a two-dimensional system in the case of a center. In this situation, the only possible resonance is for purely imaginary and opposite-sign eigenvalues of the linear part

$$\begin{aligned} \dot{x} &= \quad y + a_1 x^2 + a_2 x\, y + a_3 y^2, \\ \dot{y} &= -x + b_1 x^2 + b_2 x\, y + b_3 y^2, \end{aligned} \tag{1}$$

where we found 13 sets of parameters for which the system is integrable [3].

For the saddle case there are many resonances $\alpha : 1$ for natural values of α

$$\begin{aligned} \dot{x} &= \alpha x + a_1 x^2 + a_2 x\, y + a_3 y^2, \\ \dot{y} &= -y + b_1 x^2 + b_2 x\, y + b_3 y^2. \end{aligned} \tag{2}$$

At the resonance 1:1, i.e. at $\alpha = 1$ we got 7 cases of the integrability, and for $\alpha = 2$ also 7 sets of parameters for which the system is integrable [3].

The above results were obtained by solving algebraic systems for the parameters of the system. Each of these systems was created for a specific resonance, i.e. for a fixed natural parameter α. But the form of all these equations and their variables are the same, so the idea arises to look for a general solution to the combined system for several resonances. We created such a system by combining systems for 1:1, 2:1 and 3:1 resonances. For all sets of parameters obtained as a result of solving such a unified system, it was possible to calculate the first integrals of the system (2) for an arbitrary (symbolic) α. We found 11 sets of parameters under which the system integrates with an arbitrary α.

The first integrals for the systems discussed above were calculated using the DSolv procedure of the MATHEMATICA-11 system or manually using the Darboux method.

2. Three dimension case

First we again considered resonant cases of the system

$$\begin{aligned} \dot{x} &= \quad \alpha\, x + a_2\, x\, y + a_4\, x\, z + a_5\, y\, z, \\ \dot{y} &= -\beta\, y + b_2\, x\, y + b_4\, x\, z + b_5\, y\, z, \\ \dot{z} &= \quad -\, z + c_2\, x\, y + c_4\, x\, z + c_5\, y\, z, \end{aligned} \tag{3}$$

with natural α, β on the square table $\{1, 2, 3\} \times \{1, 2, 3\}$. In the two-dimensional case, we struggled to evaluate each integral. But here we limited ourselves to calculations only using the DSolve procedure of the MATHEMATICA 13.3.1.0 system. The results are in table 1.

N	α	β	Algebraic solutions	Integrals
8	1	1	23	19
8	1	2	16	12
8	1	3	25	19
8	2	1	57	49
8	2	2	34	29
8	2	3	43	35
9	3	1	60	51
9	3	2	63	58
10	3	3	43	38

TABLE 1

"N" here is the normal form order, "Algebraic solutions" is a number of rational solutions of the corresponding algebraic system and the "Integrals" is a number of success solutions by the MATHEMATICA.

Then we solved the unied algebraic system from these 9 systems above (329 equations), found its 10 solutions, and opened that system MATHEMATICA-13.3.1.0 solves all corresponding systems of ODEs of the form (3) except one, but the dsolve procedure of the Maple 17 calculated solutions for the 9-th case. The integrable systems for arbitary α and β are:

$$
\begin{array}{ll}
1 & \dot{x} = \alpha x + a_2 x\cdot y + a_4 x\cdot z + a_5 y\cdot z,\\
 & \dot{y} = -\beta y + b_5 y\cdot z,\\
 & \dot{z} = -z + c_5 y\cdot z;\\
2 & \dot{x} = \alpha x,\\
 & \dot{y} = -\beta y + b_2 x\cdot y + b_4 x\cdot z,\\
 & \dot{z} = -z + c_4 x\cdot z;\\
3 & \dot{x} = \alpha x + a_2 x\cdot y + a_4 x\cdot z + a_5 y\cdot z,\\
 & \dot{y} = -\beta y + a_4 y\cdot z,\\
 & \dot{z} = -z - a_2 y\cdot z;\\
4 & \dot{x} = \alpha x,\\
 & \dot{y} = -\beta y + b_2 x\cdot y,\\
 & \dot{z} = -z + c_4 x\cdot z;\\
5 & \dot{x} = \alpha x,\\
 & \dot{y} = -\beta y + b_4 x\cdot z,\\
 & \dot{z} = -z + c_4 x\cdot z;\\
6 & \dot{x} = \alpha x,\\
 & \dot{y} = -\beta y,\\
 & \dot{z} = -z + c_4 x\cdot z + c_5 y\cdot z;\\
7 & \dot{x} = \alpha x,\\
 & \dot{y} = -\beta y + b_2 x\cdot y + b_5 y\cdot z,\\
 & \dot{z} = -z;\\
8 & \dot{x} = \alpha x + a_4 x\cdot z,\\
 & \dot{y} = -\beta y + b_4 x\cdot z + a_4 y\cdot z,\\
 & \dot{z} = -z;\\
9 & \dot{x} = \alpha x + a_5 y\cdot z,\\
 & \dot{y} = -\beta y + b_2 x\cdot y,\\
 & \dot{z} = -z - b_2 x\cdot z;\\
10 & \dot{x} = \alpha x,\\
 & \dot{y} = -\beta y,\\
 & \dot{z} = -z + c_4 x\cdot z.
\end{array}
$$

Please note that a_i, b_j, c_k are free, unrelated parameters; they are arbitrary for each case separately.

3. The general three-dimension system

Finally, we considered the general case of a three-dimensional system with 20 parameters

$$\begin{aligned}\dot{x} &= \quad \alpha x + a_1 x^2 + a_2 x \cdot y + a_3 y^2 + a_4 x \cdot z + a_5 y \cdot z + a_6 z^2, \\ \dot{y} &= -\beta y + b_1 x^2 + b_2 x \cdot y + b_3 y^2 + b_4 x \cdot z + b_5 y \cdot z + b_6 z^2, \\ \dot{z} &= \quad -z + c_1 x^2 + c_2 x \cdot y + c_3 y^2 + c_4 x \cdot z + c_5 y \cdot z + c_6 z^2.\end{aligned} \tag{4}$$

Calculating the normal form up to 6th order for 4 pairs $\{\alpha, \beta\}$, i.e. for $\{1, 1\}$, $\{1, 2\}$, $\{2, 1\}$ and $\{2, 2\}$, we got a system of 121 equations with 18 parameters. We received 174 of its solutions. For 109 of them the MATHEMATICA system calculated solutions of the corresponding ODEs.

4. Chemical Kinetics Models

There are many cases of integrability of three-dimension systems, and the corresponding exact solutions can be useful in applications, for example, in problems of chemical kinetics. The explicit form of solutions allows one to study bifurcation behavior depending on the parameters of the system. This will make it possible to discover new effects in simulated systems. See, for example, the Jabotinsky-Korzukhin model [4].

$$\begin{aligned}\dot{x} &= k_1 x(C - y) - k_0 x\, z, \\ \dot{y} &= k_1 x(C - y) - k_2 y, \\ \dot{z} &= k_2 y - k_3 z.\end{aligned} \tag{5}$$

The eigenvalues of the linear part the system above are $\{C \cdot k_1, -k_2, -k_3\}$.

After diagonalizing the linear part of equation (5) takes the form (4). The question arises: under what additional conditions does the diagonalized equation (gensyst) appear among the exactly solvable cases? We found that system (5) has 5 integrable in quadratures cases if the below relations are satisfied

$$k_0 = \frac{C k_1 + 1}{C}, \quad k_2 = -C k_1, \quad k_3 = 1. \tag{6}$$

Unfortunately, the coefficients in the model (5) must be positive, so requirement (6) is not feasible in reality. But this example illustrates the possibility of discovering integrable cases of dynamical models.

References

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