Norm of orbit displacement in a problem with perturbing acceleration varying inversely with the square of the heliocentric distance

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Abstract. Let a point of zero mass move under the influence of attraction to the Sun and a small perturbing acceleration $\mathbf{P}' = \mathbf{P}/r^2$, where r is the heliocentric distance. The components of the vector $\mathbf{P}$ are assumed to be constant in one of the two reference frames: $\mathcal{O}_1$, associated with the radius vector, and $\mathcal{O}_2$, associated with the velocity vector. Here are the expressions for the Euclidean (root-mean-square over the mean anomaly) norm of displacement $\varrho^2 = ||d\mathbf{r}||^2$ in two reference frames for this problem, where $d\mathbf{r}$ represents the difference between the position vectors in the osculating and mean orbit. Using these expressions, the ϱ displacement of model asteroids with different orbital eccentricities due to the Yarkovsky effect is estimated.

Introduction

We considered the motion of the asteroid $\mathcal{A}$ under the influence of attraction to the Sun $\mathcal{S}$ and additional perturbing acceleration $\mathbf{P}'$. Let the acceleration $\mathbf{P}'$ vary inversely with the square of the r distance from $\mathcal{S}$, that is, $\mathbf{P}' = \mathbf{P}/r^2$. We introduced two orbital reference frames $\mathcal{O}_1$ and $\mathcal{O}_2$ with a origin $\mathcal{S}$. The axes for the $\mathcal{O}_1$ frame are directed along the radius vector, the transversal (perpendicular to the radius vector in the osculating plane in the direction of motion), and the binormal (directed along the angular momentum vector). The axes for the $\mathcal{O}_2$ frame are directed along the velocity vector, the main normal to the osculating orbit and the binormal.

Let the acceleration $\mathbf{P}'$ be small in comparison with the main acceleration κ^2/r^2:

$$\max \frac{|\mathbf{P}'|}{\kappa^2 r^{-2}} = \max \frac{|\mathbf{P}|}{\kappa^2} = \mu \ll 1,$$

where $\mathbf{r} = \mathcal{SA}$, $r = |\mathbf{r}|$, κ^2 is the product of the gravitation constant by the mass $\mathcal{S}$. The vector $\mathbf{P}$ has components S, T, W in the system $\mathcal{O}_1$ and components $\mathfrak{T}, \mathfrak{N}, W$ in the system $\mathcal{O}_2$. We assume that they are constant and small on the order of μ.

Sannikova and Kholshevnikov [1] applied an averaging procedure to the Euler-type equations of motion and obtained mean-elements motion equations and formulas for the transition from osculating elements to the mean ones in the first order of smallness for this problem (we neglected the second order quantities). For this problem in the $\mathcal{O}_1$ system, the paper [2] obtained the Euclidean (root mean square over the mean anomaly) displacement norm $\rho^2 = ||d\mathbf{r}||^2$, where $d\mathbf{r}$ represents the difference between the position vectors on the osculating and mean orbit. The expression for the Euclidean norm in the $\mathcal{O}_2$ reference frame is also presented below. Using these expressions, it is possible to estimate the magnitude ρ of short-period orbital disturbances arising due to the presence of a small perturbing acceleration $\mathbf{P}'$ varying inversely with the square of the heliocentric distance, e.g. due to the Yarkovsky effect.

1. Equations

The Euclidean norm of the difference between osculating and mean elements in the $\mathcal{O}_1$ is

$$\varrho_1^2 = \|d\mathbf{r}\|^2 = \frac{a^2}{\kappa^4}(A_1 S^2 + A_2 T^2 + A_3 W^2), \tag{1}$$

where

$$\begin{aligned} A_1 =& \frac{1}{2}\left(2 + 3e^2\right), \\ A_2 =& \frac{1}{(1-e^2)^2}\left(16 + \frac{3365e^2}{32} - \frac{12601e^4}{1152} - \frac{13327e^6}{2048} - \frac{226339e^8}{163840} - \mathcal{O}(e^{10})\right), \\ A_3 =& 1 - \frac{39e^2}{32} + \frac{101e^4}{576} + \frac{599e^6}{6144} + \frac{19889e^8}{307200} + \mathcal{O}(e^{10}), \end{aligned} \tag{2}$$

a is the semi-major axis and e is the eccentricity. The expressions (2) give acceptable accuracy for $e < 0.6$. More accurate expressions of the A_n functions in the form of series in powers of e and in powers of $\beta = e/(1 + \sqrt{1-e^2})$ were obtained in [2].

The Euclidean norm of the difference between osculating and mean elements in the $\mathcal{O}_2$ is

$$\varrho_2^2 = \|d\mathbf{r}\|^2 = \frac{a^2}{\kappa^4}(B_1 \mathfrak{T}^2 + B_2 \mathfrak{N}^2 + B_3 W^2). \tag{3}$$

where

$$B_1 = \frac{1}{(1-e^2)^2}\left(16 + \frac{1121e^2}{8} + \frac{10793e^4}{512} - \frac{239033e^6}{18432} - \frac{17713751e^8}{18874368} - \mathcal{O}(e^{10})\right),$$

$$B_2 = \frac{1}{(1-e^2)^2}\left(1 + \frac{29e^2}{8} - \frac{2221e^4}{288} + \frac{1907e^6}{512} - \frac{265501e^8}{491520} - \mathcal{O}(e^{10})\right),$$

$$B_3 = 1 - \frac{39e^2}{32} + \frac{101e^4}{576} + \frac{599e^6}{6144} + \frac{19889e^8}{307200} + \mathcal{O}(e^{10}). \qquad (4)$$

The expressions (4) give acceptable accuracy for $e < 0.6$. The derivation of the displacement norm in the $\mathcal{O}_2$ is being prepared for publication; more precise expressions for the B_n functions will also be given there.

Let's compare the ρ_1^2 and ρ_2^2 norms. The formulas for the main results (1) and (3) are identical up to the replacement of the components of the perturbing acceleration. In both cases, the $\|d\mathbf{r}\|^2$ norm depends only on the components of the $\mathbf{P}$ vector (positive definite quadratic form), the semimajor axis (proportional to the second power) and the eccentricity of the osculating ellipse. The $A_n(e)$ and $B_n(e)$ functions are series in even degrees of eccentricity. The $A_3(e)$ and $B_3(e)$ functions coincide, since the W component is the same for both reference systems. In the $\mathcal{O}_1$ frame the $A_1(e)$ function is a polynomial of the second degree, while in the $\mathcal{O}_2$ system $B_1(e)$ is an infinite series, $A_2(e)$ and $B_2(e)$ are series in both systems. Since at zero eccentricity the $(-\mathfrak{N}, \mathfrak{T}, W)$ trihedron is identical to the (S, T, W) trihedron, then $A_1(0) = B_2(0)$, $A_2(0) = B_1(0)$ and $A_3(0) = B_3(0)$, that is, the free terms of (2) and (4) coincide, as it should be.

2. Application

e	$\mathfrak{T}$, 10^{-14} AU^3/day^2	$\mathfrak{N}$, 10^{-14} AU^3/day^2	ϱ_2, m	ϱ_1, m
0.001	-5.10168	-9.91079	129.185	129.185
0.01	-5.10155	-9.91054	129.245	129.231
0.10	-5.08887	-9.88585	135.127	133.848
0.20	-5.04976	-9.80969	152.479	147.865
0.30	-4.98212	-9.67805	180.585	171.674
0.40	-4.88179	-9.48280	219.968	206.987
0.50	-4.74156	-9.20998	273.527	258.152
0.60	-4.54897	-8.83547	348.406	335.067
0.70	-4.28099	-8.31451	461.304	461.827
0.80	-3.88832	-7.55138	658.382	711.424
0.90	-3.22864	-6.26976	1136.522	1448.588
0.99	-1.53792	-2.98595	5562.831	14545.945

TABLE 1. Tangential $\mathfrak{T}$ and normal $\mathfrak{N}$ components, the ϱ_1 and ϱ_2 displacements, calculated at different eccentricities e

The article [3] considers model objects with different orbital eccentricities from 0 to 0.99, and other orbital and thermophysical characteristics, like asteroid Bennu, and finds the mean-orbital values of the $\mathbf{P}$ vector components in the $\mathcal{O}_1$ and $\mathcal{O}_2$ systems. Turning to the results [3], let us calculate the ϱ_1 and ϱ_2

orbit displacements for these model objects. The following constants were used in the calculations: $1\,\mathrm{AU} = 1.495978707 \times 10^{11}\,\mathrm{m}$, $\kappa^2 = 1.32712440041279419 \times 10^{20}\,\mathrm{m^3 s^{-2}}$, $1\,\mathrm{day} = 86400\,\mathrm{s}$. For all cases $a = 1.126391025894812$ AU, $S = 9.91079 \times 10^{-14}$ AU^3/day^2, $T = -5.10168 \times 10^{-14}$ AU^3/day^2, $W = 0$. The table 1 contains the other initial data and calculation results.

From the table 1 it is clear that as e increases, the magnitude of periodic disturbances caused by the Yarkovsky effect increases, although the modulo values of the $\mathfrak{T}$ and $\mathfrak{N}$ components decrease. In the $\mathcal{O}_1$ system, the S and T components do not depend on e, but the increase in ϱ_1 at high e is more pronounced than in the $\mathcal{O}_2$ system. This may indicate an overestimation of the short-period orbital disturbances for objects in highly elliptical orbits when it is calculated in the $\mathcal{O}_1$ system.

In general, at low perturbing acceleration characteristic of the Yarkovsky effect, the displacement of the osculating orbit relative to the mean one is small and can be neglected, taking into account only the secular drifts of the orbital elements, as was shown in [2].

Conclusion

Expressions for the Euclidean (root mean square over the mean anomaly) norm of the difference between osculating and mean elements are represent in two orbital frames of reference: $\mathcal{O}_1$, associated with the radius vector, and $\mathcal{O}_2$, associated with the velocity vector. The short-period orbit disturbances of model asteroids with different orbital eccentricities due to the Yarkovsky effect is estimated.

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References

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