In memoriam of Konstantin Kholshevnikov

I thank the organizers of the conference for the invitation and the opportunity to speak.
Konstantin Vladislavovich was my scientific supervisor in graduate school, and for me he has always been and remains a Teacher with a capital letter.
Despite the fact that he is no longer with us, I continue to learn from him through his books and articles and still draw inspiration from his ideas.
Here I will present material that is a continuation of the work begun jointly with Konstantin Vladislavovich.

The photo was taken at the V.P. Engelhardt Observatory at a conference in honor of the 200th anniversary of the Kazan University Astronomy Department (July 10, 2010)

Norm of orbit displacement in a problem with perturbing acceleration varying inversely with the square of the heliocentric distance

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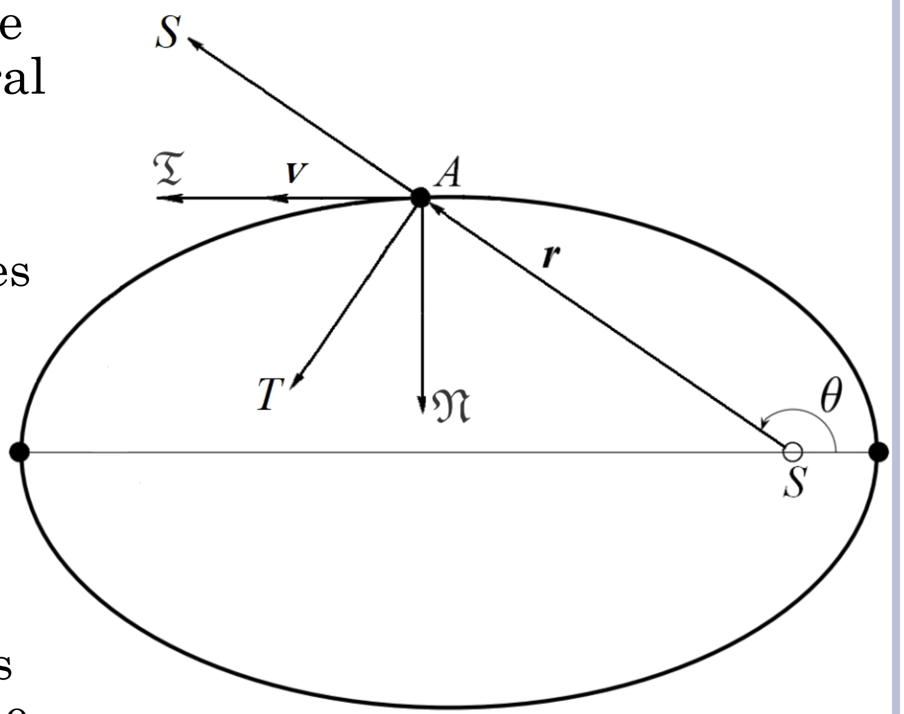
Statement of the Problem

- A zero-mass point moves under the influence of attraction to the central body and a small perturbing acceleration $\boldsymbol{P'}=\boldsymbol{P}/r^2$
- Orbital reference frame O_1: with axes directed along the radius vector, the transversal and the angular momentum vector

$$\boldsymbol{P}\,(S,\ T,\ W):$$
$$S,\ T,\ W = const;$$

- Orbital reference frame O_2 with axes directed along the velocity vector, the main normal and the angular momentum vector

$$\boldsymbol{P}\,(\mathfrak{T},\ \mathfrak{N},\ W)$$
$$\mathfrak{T},\ \mathfrak{N},\ W = const;$$



The components of the perturbing acceleration are considered to be constant and small quantities of the order of μ: $\max \frac{|\mathbf{P}'|}{\varkappa^2 r^{-2}} = \max \frac{|\mathbf{P}|}{\varkappa^2} = \mu \ll 1.$

Equations of motion in mean elements in a first approximation in a small parameter

$$\dot{n} = -\frac{3n^2}{\varkappa^2\eta^2}T = -\frac{6n^2}{\pi\varkappa^2\eta^2}[2\mathbf{E}(e) - \eta^2\mathbf{K}(e)]\mathfrak{T},$$

$$\dot{e} = \frac{ne}{\varkappa^2(1+\eta)}T = \frac{4n}{\pi\varkappa^2 e}\left[\mathbf{E}(e) - \eta^2\mathbf{K}(e)\right]\mathfrak{T},$$

$$\dot{i} = -\frac{ne\cos\sigma}{\varkappa^2\eta(1+\eta)}W,$$

$$\dot{\Omega} = -\frac{ne\sin\sigma}{\varkappa^2\eta(1+\eta)\sin i}W,$$

$$\dot{\sigma} = \frac{ne\sin\sigma\cot i}{\varkappa^2\eta(1+\eta)}W = \frac{2n}{\pi\varkappa^2}\mathbf{K}(e)\mathfrak{N} + \frac{ne\sin\sigma\cot i}{\varkappa^2\eta(1+\eta)}W,$$

$$\dot{M} = n - \frac{2n}{\varkappa^2}S = n + \frac{2n\eta}{\pi\varkappa^2}\mathbf{K}(e)\mathfrak{N}.$$

n – mean motion,
e – eccentricity,
i – inclination,
Ω – longitude of the ascending node,
σ – argument of pericenter,
M – mean anomaly

$\eta = \sqrt{1-e^2}$,

semi-major axis $a = \varkappa^{2/3}n^{-2/3}$

The transition from osculating elements to mean elements: $\epsilon_n = \bar{\epsilon}_n - \delta\epsilon_n$

$\bar{\epsilon}_n$ are the osculating elements , ϵ_n are the mean elements

$\delta\epsilon_n$ are the change-of-variable functions

T.N. Sannikova, K.V. Kholshevnikov, The Averaged Equations of Motion in the Presence of an Inverse-Square Perturbing Acceleration // Astronomy Reports — 2019. — V.63, No.5 — p. 420-432 (2019).

NORM OF DIFFERENCE BETWEEN OSCULATING AND MEAN ELEMENTS

If the perturbing forces are small, then the osculating orbit deviates slightly from the mean one. The difference $d\mathbf{r}$ in positions of the celestial body on the mean and osculating orbits is a quasiperiodic function of time. The $\|d\mathbf{r}\|^2$ Euclidean (root mean square for the mean anomaly) norm of the displacement of the osculating orbit relative to the mean orbit allows us to estimate the magnitude of short-period disturbances arising as a result of forces inversely proportional to the square of the distance from the Sun (for example, the Yarkovsky effect, the pressure of sunlight), and to decide on the need to take them into account or the possibility of limiting ourselves to only secular drifts, which are given by the averaged equations of motion.

Formulas for calculating the displacement $\varrho = \sqrt{\|d\mathbf{r}\|^2}$, which characterizes the magnitude of the deviation of the osculating orbit from the mean orbit, were found for orbital reference systems O_1, associated with the radius vector, and O_2, associated with the velocity vector.

Norm of difference between osculating and mean elements

According to [Batmunkh et al., 2016], the difference between the osculating and mean radius vectors can be expressed through the differences in the orbital elements:

$$(d\mathbf{r})^2 = \delta r^2 + r^2(\delta u + \cos i \delta\Omega)^2 + r^2(\sin u \delta i - \sin i \cos u \delta\Omega)^2,$$

where u is the latitude argument,

$$\delta r = \frac{r}{a}\delta a + \frac{a^2}{r}(e - \cos E)\delta e + \frac{a^2}{r} e \sin E\, \delta M,$$

$$r(\delta u + \cos i \delta\Omega) = \frac{a^2}{\eta r}\left(2 - e^2 - e\cos E\right)\sin E \delta e + r\delta\sigma + r\cos i\,\delta\Omega + \frac{a^2\eta}{r}\delta M,$$

$$r(\sin u \delta i - \sin i \cos u \delta\Omega) = a\left[(\cos E - e)\sin\sigma + \eta \sin E \cos\sigma\right]\delta i -$$

$$- a \sin i\left[(\cos E - e)\cos\sigma - \eta\sin E \sin\sigma\right]\delta\Omega.$$

The Euclidean (root mean square for the mean anomaly) norm of the displacement of the osculating orbit relative to the mean orbit is calculated using the formula

$$\varrho^2 = \|d\mathbf{r}\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(d\mathbf{r})^2\, dM = \frac{1}{2\pi}\int_{-\pi}^{\pi}(d\mathbf{r})^2(1 - e\cos E)\, dE = \frac{1}{2\pi}\int_{-\pi}^{\pi}(d\mathbf{r})^2\frac{r}{a}\, dE.$$

N. Batmunkh, T.N. Sannikova, K.V. Kholshevnikov, V.Sh. Shaidulin. The Norm of the Position Shift of a Celestial Body upon Variation of its Orbit // Astronomy Reports — 2016 —V.60, No.3 — p. 366-373.

The change-of-variable functions in O_1

$$\delta n = \frac{3neS}{\eta^2\varkappa^2}\left(\frac{a}{r}(\cos E - e) + e\right) - \frac{3nT}{\eta^2\varkappa^2}\left(e\sin E(1+\frac{a}{r}\eta) + 2A(\beta,E)\right),$$

$$\delta a = -\frac{2aeS}{\eta^2\varkappa^2}\left(\frac{a}{r}(\cos E - e) + e\right) + \frac{2aT}{\eta^2\varkappa^2}\left(e\sin E(1+\frac{a}{r}\eta) + 2A(\beta,E)\right),$$

$$\delta e = -\frac{S}{\varkappa^2}\left(\frac{a}{r}(\cos E - e) + e\right) + \frac{T}{e\varkappa^2}\left(e\sin E(1-\eta+\frac{a}{r}\eta) + 2A(\beta,E)\right),$$

$$\delta i = \frac{W}{e\eta\varkappa^2}\Big(\cos\sigma\big[e\sin E(\eta-1) + 2\eta A(\beta,E)\big] + \eta\sin\sigma\big[1-\eta-L(\beta,E)\big]\Big),$$

$$\delta\Omega = \frac{W}{e\eta\varkappa^2\sin i}\Big(\sin\sigma\big[e\sin E(\eta-1) + 2\eta A(\beta,E)\big] - \eta\cos\sigma\big[1-\eta-L(\beta,E)\big]\Big),$$

$$\delta\sigma = -\frac{\eta S}{e\varkappa^2}\frac{a}{r}\sin E - \frac{T}{e^2\varkappa^2}\left(e\frac{a}{r}(\cos E - e) + e^2 + 1 - \eta - L(\beta,E)\right) - \delta\Omega\cos i,$$

$$\delta M = \frac{S}{\varkappa^2}\left(e\sin E + \frac{\eta^2}{e}\frac{a}{r}\sin E\right) + \frac{T}{\eta^2\varkappa^2}\left[\frac{(\eta+2)\eta^3}{\eta+1} + \frac{\eta^3}{e}\frac{a}{r}(\cos E - e) - \frac{3}{4}e^2\cos 2E + \right.$$
$$\left. +3e(\eta+1)\left(\frac{e}{2}+\cos E\right) - \frac{\eta^3}{e^2}L(\beta,E) + \frac{3\beta(2+\beta^2)}{1+\beta^2}\left(\frac{e}{2}+\cos E\right) - \frac{6}{1+\beta^2}S(\beta,E)\right],$$

$r = a(1-e\cos E)$, $\beta = e/(1+\eta)$, $\eta = \sqrt{1-e^2}$, E is the eccentric anomaly

$$A(\beta,E) = \arctan\frac{\beta\sin E}{1-\beta\cos E}, \quad L(\beta,E) = \ln\left(1 - 2\beta\cos E + \beta^2\right),$$

$$S(\beta,E) = \sum_{n=2}^{\infty}\frac{n+1-(n-1)\beta^2}{n^2(n^2-1)}\beta^n\cos nE.$$

T.N. Sannikova. Displacement Norm in the Presence of an Inverse-Square Perturbing Acceleration in the Reference Frame Associated with the Radius Vector // Astron. Rep. — 2024 — V. 68, No.3 — p. 331—343

The difference between the osculating and mean radius vectors in O_1

$$(d\mathbf{r})^2 = \delta r^2 + r^2(\delta u + \cos i\delta\Omega)^2 + r^2(\sin u\delta i - \sin i\cos u\delta\Omega)^2$$

$$\delta r = \frac{Sa^3}{4\varkappa^2 r^2}\Phi_1 + \frac{Ta^3}{\varkappa^2 r^2}\Phi_2, \quad r(\delta u + \cos i\delta\Omega) = \frac{Ta^3}{\varkappa^2 r^2}\Phi_3, \quad r(\sin u\delta i - \sin i\cos u\delta\Omega) = \frac{Wa}{\varkappa^2 e}\Phi_4$$

$$\Phi_1 = 2(2+3e^2) - 3e(4+e^2)\cos E + 6e^2\cos 2E - e^3\cos 3E,$$

$$\begin{aligned}\Phi_2 = {} & \frac{e\sin E}{4\eta^3(1+\eta)^2}\left(44(1+\eta) - e^2(26+6\eta) - 3e^4(8+5\eta) + 6e^6\right) - \\ & - \frac{e^2\sin 2E}{8\eta^3(1+\eta)^2}\left(24(1+\eta) + 6e^2\eta - e^4(30+17\eta) + 6e^6\right) - \\ & - \frac{e^3\sin 3E}{4\eta^3(1+\eta)^2}\left(4(1+\eta) - e^2(4+\eta)\right) - \frac{e^4\sin 4E}{16\eta^2} + \\ & + \frac{A(\beta,E)}{e\eta^2}\Big(e(7+3e^2) - (2+12e^2+e^4)\cos E + e(1+5e^2)\cos 2E - e^4\cos 3E\Big) - \\ & - \left(\frac{\eta}{e}\sin E - \frac{\eta}{2}\sin 2E\right)L(\beta,E) - \frac{3e(1+\eta)}{2\eta^2}(2\sin E - e\sin 2E)S(\beta,E),\end{aligned}$$

$$\begin{aligned}\Phi_3 = {} & \frac{16(1+\eta) - 4e^2(5+8\eta) + 9e^4}{8\eta(1+\eta)} + \frac{e\cos E}{8\eta(1+\eta)^2}\left(128(1+\eta) - e^2(98+38\eta) - e^4\right) - \\ & - \frac{e^2\cos 2E}{2\eta(1+\eta)^2}\left(28(1+\eta) - e^2(18+5\eta)\right) + \frac{e\cos 3E}{8\eta}\left(4(1-\eta) + e^2(3+4\eta)\right) - \\ & - \frac{e^2\cos 4E}{8\eta}(1-\eta) + \frac{A(\beta,E)}{2e\eta}\Big(\left(8-3e^2\right)\sin E - 2e\left(3-e^2\right)\sin 2E + e^2\sin 3E\Big) + \\ & + \left(\frac{5}{2} - \frac{1}{4e}\left(8+7e^2\right)\cos E + \frac{3}{2}\cos 2E - \frac{e}{4}\cos 3E\right)L(\beta,E) - \frac{3(1+\eta)}{\eta}(1-e\cos E)\,S(\beta,E),\end{aligned}$$

$$\Phi_4 = \frac{(1-\eta)}{2}(-3e + 2\cos E + e\cos 2E) + 2\eta A(\beta,E)\sin E + (e-\cos E)L(\beta,E).$$

T.N. Sannikova. Displacement Norm in the Presence of an Inverse-Square Perturbing Acceleration in the Reference Frame Associated with the Radius Vector // Astron. Rep. — 2024 — V. 68, No.3 — p. 331—343

Norm of difference between osculating and mean elements in O_1

$$(d\mathbf{r})^2 = \frac{S^2a^6}{16\varkappa^4r^4}\Phi_1^2 + \frac{STa^6}{2\varkappa^4r^4}\Phi_1\Phi_2 + \frac{T^2a^6}{\varkappa^4r^4}(\Phi_2^2+\Phi_3^2) + \frac{W^2a^2}{\varkappa^4e^2}\Phi_4^2$$

The denominators of the first three terms contain r^4, but after squaring and reducing like terms, the resulting expressions are reduced by r^3:

$$(d\mathbf{r})^2 = \frac{S^2a^2}{16\varkappa^4}\frac{a}{r}\Psi_1 + \frac{T^2a^2}{\varkappa^4}\frac{a}{r}\Psi_2 + \frac{W^2a^2}{\varkappa^4e^2}\Psi_3 + \frac{STa^2}{\varkappa^4}\frac{a}{r}\Psi_4,$$

$$\Psi_1 = 8(2+3e^2) - 12e(4+e^2)\cos E + 24e^2\cos 2E - 4e^3\cos 3E,$$

$$\Psi_2 = \psi_{21}+\psi_{22}+\psi_{23}+\psi_{24}+\psi_{25}+\psi_{26}+\psi_{27}+\psi_{28}+\psi_{29}+\psi_{2,10},$$

$$\Psi_3 = \psi_{31}+\psi_{32}+\psi_{33}+\psi_{34}+\psi_{35}+\psi_{36},$$

$$\Psi_4 = \psi_{41}+\psi_{42}+\psi_{43}+\psi_{44}.$$

The expressions ψ_{ii} have the form of a trigonometric polynomial with coefficients depending on the eccentricity ($\sum a_k(e)\sin kE,\ \sum b_k(e)\cos kE$), or the product of a trigonometric polynomial by functions of the form

$$A(\beta,E) = \arctan\frac{\beta\sin E}{1-\beta\cos E},\ \ L(\beta,E) = \ln\left(1-2\beta\cos E+\beta^2\right),\ \ S(\beta,E) = \sum_{n=2}^{\infty}\frac{n+1-(n-1)\beta^2}{n^2(n^2-1)}\beta^n\cos nE$$

or their combinations: $A(\beta,E)^2$, $L(\beta,E)^2$, $S(\beta,E)^2$, $A(\beta,E)L(\beta,E)$, $A(\beta,E)S(\beta,E)$, $L(\beta,E)S(\beta,E)$.
The remaining r's in the denominators will be cancelled out when calculating the root mean square norm from the mean anomaly.

$$\varrho^2 = \|d\mathbf{r}\|^2 = \frac{S^2a^2}{16\varkappa^4}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi_1\,dE + \frac{T^2a^2}{\varkappa^4}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi_2\,dE + \frac{W^2a^2}{\varkappa^4e^2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi_3\frac{r}{a}\,dE + \frac{STa^2}{\varkappa^4}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi_4\,dE$$

Calculating of ϱ^2 is reduced to finding integrals of the Ψ_1, Ψ_2, Ψ_3, Ψ_4 functions.

INTEGRATION

The function $\Psi_4 = \psi_{41} + \psi_{42} + \psi_{43} + \psi_{44}$,

$$\psi_{41} = \frac{e}{8\eta^3(\eta+1)^2}\Big(4\left[44(\eta+1) - e^2(6\eta+26) - 3e^4(5\eta+8) + 6e^6\right]\sin E -$$

$$- 2e\left[24(\eta+1) + 6e^2\eta - e^4(17\eta+30) + 6e^6\right]\sin 2E -$$

$$- 4e^2\left[4(\eta+1) - e^2(\eta+4)\right]\sin 3E - e^3\left[2(\eta+1) - e^2(\eta+2)\right]\sin 4E\Big),$$

$$\psi_{42} = \frac{2}{e\eta^2}A(\beta,E)\left(e[7+3e^2] - [2+12e^2+e^4]\cos E + e[1+5e^2]\cos 2E - e^4\cos 3E\right),$$

$$\psi_{43} = -\frac{\eta}{e}L(\beta,E)\left(2\sin E - e\sin 2E\right),$$

$$\psi_{44} = -\frac{3e(\eta+1)}{\eta^2}S(\beta,E)\left(2\sin E - e\sin 2E\right).$$

is odd, since sin kE and $A(\beta, E)$ are odd, and $L(\beta, E)$ and $S(\beta, E)$ are even, therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi_4\,dE = 0.$$

Integration of even functions

When integrating functions containing trigonometric polynomials, $A(\beta,E)$, $L(\beta,E)$ or $S(\beta,E)$, we use the integrals given in the books:

K.V. Kholshevnikov, V.B. Titov. Two-Body-Problem (SPbGU, St. Petersburg, 2007):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dE = 1,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos kE\, dE = 0,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r}{a}\cos E\, dE = -\frac{e}{2},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r}{a}\cos kE\, dE = 0 \text{ при } k \geq 2$$

I.S. Gradstein and I.M. Ryzhik. Tables of Integrals, Series and Products (BKhV-Peterburg, St. Petersburg, 2011; Elsevier Inc., Burlington, 2007):

$$\int_0^{\pi} \arctan\left(\frac{\beta\sin E}{1-\beta\cos E}\right)\sin kE\, dE = \frac{\pi}{2k}\beta^k \text{ при } \beta^2 < 1,$$

$$\int_0^{\pi} \ln\left(1 - 2\beta\cos E + \beta^2\right) dE = 0 \text{ при } \beta^2 \leq 1,$$

$$\int_0^{\pi} \ln\left(1 - 2\beta\cos E + \beta^2\right)\cos kE\, dE = -\frac{\pi}{k}\beta^k \text{ при } \beta^2 < 1$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} A(\beta,E)\sin kE\, dE = \frac{1}{2\pi}\int_{-\pi}^{\pi}\arctan\left(\frac{\beta\sin E}{1-\beta\cos E}\right)\sin kE\, dE = \frac{\beta^k}{2k}, \quad k \geq 1$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} L(\beta,E)\, dE = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(1 - 2\beta\cos E + \beta^2\right) dE = 0,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} L(\beta,E)\cos kE\, dE = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(1 - 2\beta\cos E + \beta^2\right)\cos kE\, dE = -\frac{\beta^k}{k}$$

Integration of functions with $A(\beta,E)^2$ or $L(\beta,E)^2$

Integration of functions containing $A(\beta,E)^2$ or $L(\beta,E)^2$ was performed using series expansions [Gradshteyn and Ryzhik, 2011]:

$$A(\beta,E) = \arctan\left(\frac{\beta\sin E}{1-\beta\cos E}\right) = \sum_{n=1}^{\infty}\frac{\beta^n}{n}\sin nE, \quad L(\beta,E) = \ln\left(1-2\beta\cos E+\beta^2\right) = -2\sum_{n=1}^{\infty}\frac{\beta^n}{n}\cos nE,$$

and transformation of products of trigonometric functions into sums:

$$\sum_{n=0}^{\infty} a_n \cos nE \cos kE = \frac{1}{2}\sum_{n=0}^{\infty} a_n[\cos(n-k)E + \cos(n+k)E],$$

$$\sum_{n=0}^{\infty} a_n \sin nE \cos kE = \frac{1}{2}\sum_{n=0}^{\infty} a_n[\sin(n-k)E + \sin(n+k)E],$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} A(\beta,E)^2 \cos kE\, dE = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\beta,E)\sum_{n=1}^{\infty}\frac{\beta^n}{n}\sin nE\ \cos kE\, dE =$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\beta^n}{2n}\int_{-\pi}^{\pi} A(\beta,E)[\sin(n-k)E + \sin(n+k)E]\, dE$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} L(\beta,E)^2 \cos kE\, dE = \frac{1}{2\pi}\int_{-\pi}^{\pi} L(\beta,E)\left(-2\sum_{n=1}^{\infty}\frac{\beta^n}{n}\cos nE\right)\cos kE\, dE =$$

$$= -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\beta^n}{n}\int_{-\pi}^{\pi} L(\beta,E)[\cos(n-k)E + \cos(n+k)E]\, dE.$$

Integration of functions with $A(\beta,E)S(\beta,E)$ or $L(\beta,E)S(\beta,E)$

The function $S(\beta,E)$ is a series

$$S(\beta,E)=\sum_{n=2}^{\infty}\frac{n+1-(n-1)\beta^2}{n^2(n^2-1)}\beta^n\cos nE\,,$$

therefore, functions containing products $A(\beta,E)S(\beta,E)$ or $L(\beta,E)S(\beta,E)$ are reduced to table integrals by transforming products of trigonometric functions into sums:

$$\sum_{n=0}^{\infty}a_n\cos nE\cos kE=\frac{1}{2}\sum_{n=0}^{\infty}a_n[\cos(n-k)E+\cos(n+k)E],$$

$$\sum_{n=0}^{\infty}a_n\sin nE\cos kE=\frac{1}{2}\sum_{n=0}^{\infty}a_n[\sin(n-k)E+\sin(n+k)E],$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}A(\beta,E)S(\beta,E)\sin kE\,dE=\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{n+1-(n-1)\beta^2}{2n^2(n^2-1)}\beta^n\int_{-\pi}^{\pi}A(\beta,E)[\sin(n+k)E-\sin(n-k)E]\,dE$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}L(\beta,E)S(\beta,E)\cos E\,dE=\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{n+1-(n-1)\beta^2}{2n^2(n^2-1)}\beta^n\int_{-\pi}^{\pi}L(\beta,E)[\cos(n-k)E+\cos(n+k)E]\,dE$$

INTEGRATION OF FUNCTIONS WITH $A(\beta,E)L(\beta,E)$,

$$A(\beta,E) = \arctan\left(\frac{\beta\sin E}{1-\beta\cos E}\right) = \sum_{n=1}^{\infty}\frac{\beta^n}{n}\sin nE,\quad L(\beta,E) = \ln\left(1-2\beta\cos E+\beta^2\right) = -2\sum_{n=1}^{\infty}\frac{\beta^n}{n}\cos nE,$$

The functions containing $A(\beta,E)L(\beta,E)$ are

$$\psi_{28} = -\frac{1}{e^2\eta}A(\beta,E)L(\beta,E)\left(3e\sin E+6\eta^2\sin 2E-e\sin 3E\right),\quad \psi_{36} = 2\eta A(\beta,E)L(\beta,E)(2e\sin E-\sin 2E)$$

There are three variants of the integral here: $\frac{1}{2\pi}\int_{-\pi}^{\pi}A(\beta,E)L(\beta,E)\sin kE\,dE$ при $k=1,2,3.$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}A(\beta,E)L(\beta,E)\sin kE\,dE = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\frac{\beta^n}{n}\sin nE\left(-2\sum_{m=1}^{\infty}\frac{\beta^m}{m}\cos mE\right)\sin kE\,dE =$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\beta^n}{n}\frac{\beta^m}{m}\sin nE\cos mE\,\sin kE\,dE.$$

The product of the series $\sum_{n=1}^{\infty}a_n\sin nE$ and $\sum_{m=1}^{\infty}a_m\cos mE$ is

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_na_m\sin nE\cos mE = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_na_m[\sin(n-m)E+\sin(n+m)E].$$

Multiplying by sin kE gives: $s_k = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_na_m\sin nE\cos mE\,\sin kE =$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_na_m[\cos(n-m-k)E-\cos(n-m+k)E+\cos(n+m-k)E-\cos(n+m+k)E].$$

The integral of s_k is not equal to zero only if the arguments of the cosines are zero, which is possible if $n-m-k=0$, $n-m+k=0$ or $n+m-k=0$. Therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi}s_k\,dE = \frac{1}{2}\left(\sum_{n=1}^{k-1}(a_na_{k-n}-a_na_{n+k}) - a_ka_{2k} + \sum_{n=k+1}^{\infty}(a_na_{n-k}-a_na_{n+k})\right)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}A(\beta,E)L(\beta,E)\sin E\,dE = -\frac{1}{\pi}\int_{-\pi}^{\pi}s_1\,dE = 0,\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}A(\beta,E)L(\beta,E)\sin 2E\,dE = -\frac{1}{\pi}\int_{-\pi}^{\pi}s_2\,dE = -\frac{a_1^2}{2} = -\frac{\beta^2}{2},$$

$$a_n = \beta^n/n,\ a_m = \beta^m/m.\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}A(\beta,E)L(\beta,E)\sin 3E\,dE = -\frac{1}{\pi}\int_{-\pi}^{\pi}s_3\,dE = -a_1a_2 = -\frac{\beta^3}{2}.$$

Integration of functions with $S(\beta,E)^2$

The function $S(\beta,E)$ is a series $S(\beta,E)=\sum_{n=2}^{\infty}\frac{n+1-(n-1)\beta^2}{n^2(n^2-1)}\beta^n\cos nE$

$S(\beta,E)^2$ is included in the expression for $\psi_{27}=-\frac{9S(\beta,E)^2}{\eta^5}(1+e\cos E)\left(e^2(\eta+2)-2(\eta+1)\right)$

When integrating ψ_{27}, we take into account the transformation of the products of cosines of multiple arguments into a sum:

$$\sum_{n=0}^{\infty}a_n\cos nE\cos kE=\frac{1}{2}\sum_{n=0}^{\infty}a_n[\cos(n-k)E+\cos(n+k)E]$$

For a trigonometric series of the form $s=\sum_{n=0}^{\infty}a_n\cos nE$ the following relations are valid:

$$s^2=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_na_k\cos nE\cos kE=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_na_k[\cos(n-k)E+\cos(n+k)E],$$

$$s^2(1+e\cos E)=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_na_k[\cos(n-k)E+\cos(n+k)E]+$$
$$+\frac{e}{4}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_na_k[\cos(n-k-1)E+\cos(n-k+1)E+\cos(n+k-1)E+\cos(n+k+1)E].$$

Integrating the latter is equivalent to leaving only free terms in the trigonometric series:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}s^2(1+e\cos E)\,dE=a_0^2+ea_0a_1+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+ea_na_{n+1}).$$

As a result: $\frac{1}{2\pi}\int_{-\pi}^{\pi}S(\beta,E)^2(1+e\cos E)\,dE=\frac{1}{2}\sum_{n=2}^{\infty}(a_n^2+ea_na_{n+1}),\quad a_n=\frac{n+1-(n-1)\beta^2}{2n^2(n^2-1)}\beta^n.$

Norm of difference between osculating and mean elements in O_1

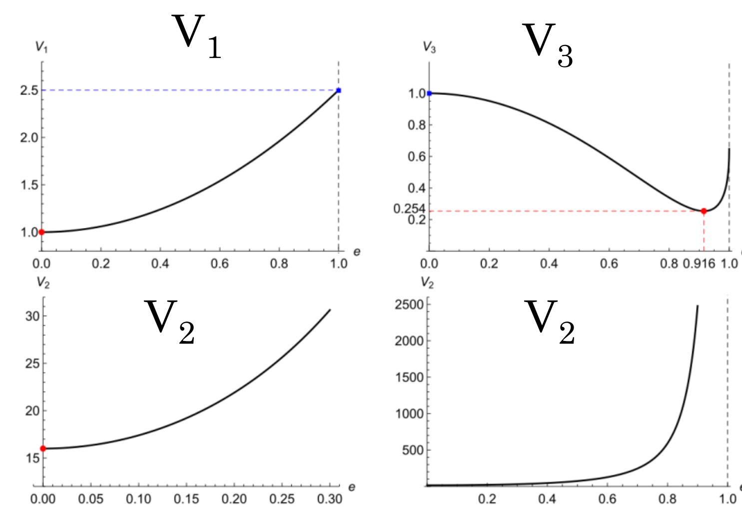
We substitute the results of integration into the original expression, reduce similar terms, and as a result we obtain the norm in the form

$$\varrho^2 = \frac{a^2}{\varkappa^4}\left(V_1 S^2 + V_2 T^2 + V_3 W^2\right),$$

$$V_1 = \frac{1 + 8\beta^2 + \beta^4}{(\beta^2+1)^2},$$

$$V_2 = \frac{1}{(1-\beta^2)^4(\beta^2+1)^2}\left(16 + \frac{4133}{8}\beta^2 + \frac{125819}{72}\beta^4 + \frac{598249}{288}\beta^6 + \frac{8028469}{5760}\beta^8 + \right.$$
$$+ \frac{190263323}{360000}\beta^{10} + \frac{1366999573}{11760000}\beta^{12} + \frac{2497264733}{345744000}\beta^{14} + \frac{2619693301}{49787136000}\beta^{16} +$$
$$\left. + \frac{247809799}{224042112000}\beta^{18} + \frac{6893993549}{90363651840000}\beta^{20} + \frac{25909767071}{2733500468160000}\beta^{22} + \mathcal{O}(\beta^{24})\right),$$

$$V_3 = \frac{1}{(\beta^2+1)^2}\left(1 - \frac{23}{8}\beta^2 + \frac{137}{36}\beta^4 + \frac{181}{288}\beta^6 + \frac{13}{400}\beta^8 + \frac{1}{720}\beta^{10} + \frac{29}{176400}\beta^{12} + \right.$$
$$\left. + \frac{1}{31360}\beta^{14} + \frac{53}{6350400}\beta^{16} + \mathcal{O}(\beta^{18})\right). \quad \eta = \sqrt{1-e^2}, \quad \beta = e/(1+\eta)$$



Behavior of functions V_1, V_2 and V_3 on interval $e \in [0,1)$ is:

$$\min(V_1) = 1 \text{ for } e = 0, \quad \max(V_1) = 2.5 \text{ for } e = 1, \quad V_1 \text{ monotonically increases;}$$

$$\min(V_2) = 16 \text{ for } e = 0, \quad V_2 \to \infty \text{ for } e \to 1, \quad V_2 \text{ monotonically increasing;}$$

$$\min(V_3) = 0.253528 \text{ for } e \approx 0.91557, \quad \max(V_3) = 1 \text{ for } e = 0.$$

Thus $V_k > 0$, hence the norm ϱ^2 is always positive, ϱ is a real number. The dependence of the V_1, V_2 and V_3 functions on e is shown in the figure.

The change-of-variable functions in O_2

$$\delta n = -\frac{6n}{\varkappa^2(1-e)}\left[\mathcal{F}_2\left(\frac{\theta}{2},k\right) - \frac{1}{\pi}\mathbf{E}\left(k\right)M\right]\mathfrak{T}, \qquad \delta a = \frac{4a}{\varkappa^2(1-e)}\left[\mathcal{F}_2\left(\frac{\theta}{2},k\right) - \frac{1}{\pi}\mathbf{E}\left(k\right)M\right]\mathfrak{T},$$

$$\delta e = \frac{4}{\varkappa^2}\left\{\mathcal{F}_1\left(\frac{\theta}{2},k\right) - \frac{1}{\pi}\mathbf{K}\left(k\right)M - \frac{2}{(1+e)}\left[\mathcal{F}_3\left(\frac{\theta}{2},k\right) - \frac{1}{\pi}\mathbf{D}\left(k\right)M\right]\right\}\mathfrak{T} +$$
$$+ \frac{2\eta}{\varkappa^2 e}\left[\operatorname{arctg}\frac{\vartheta}{\eta} - \frac{\pi}{4} - \frac{1}{\pi}[\eta^2\mathbf{K}(e) - \mathbf{E}(e)]\right]\mathfrak{N},$$

$$\delta i = \frac{1}{\varkappa^2\eta e}\left\{\cos\sigma\left[\eta(\theta - M) - (E - M)\right] + \eta\sin\sigma\left[\ln(1 + e\cos\theta) + 1 - \eta - \ln\frac{2\eta^2}{1+\eta}\right]\right\}W\,,$$

$$\delta\Omega = \frac{1}{\varkappa^2\eta e\sin i}\left\{\sin\sigma\left[\eta(\theta - M) - (E - M)\right] - \eta\cos\sigma\left[\ln(1 + e\cos\theta) + 1 - \eta - \ln\frac{2\eta^2}{1+\eta}\right]\right\}W\,,$$

$$\delta\sigma = -\frac{2}{\varkappa^2 e^2}\left[\vartheta - \frac{2\eta}{\pi}\mathbf{E}(e)\right]\mathfrak{T} +$$
$$+ \frac{1}{\varkappa^2}\left[\frac{1}{\eta}\left(\mathcal{F}_1\left(E + \frac{\pi}{2}, e\right) - \mathbf{K}\left(e\right)\left(1 + \frac{2}{\pi}M\right)\right) + \frac{1}{e^2}\ln\frac{e\sin E + \sqrt{1 - e^2\cos^2 E}}{\eta}\right]\mathfrak{N} - \delta\Omega\cos i\,,$$

$$\delta M = \frac{2}{\varkappa^2(1-e)}\left\{2(1-e)\left[\operatorname{arctg}\frac{\vartheta}{\eta} - \frac{\pi}{4} + \frac{2}{\pi}\mathbf{E}(e) - \frac{\eta^2}{\pi}\mathbf{K}(e) + \frac{1}{e^2}\left(\frac{\eta}{2}\vartheta - \frac{1}{\pi}\mathbf{E}(e)\right)\right] +\right.$$
$$\left. + \frac{3\mathbf{E}(k)}{\pi}\left[e\left(\cos E + \frac{e}{2}\right) - \frac{e^2}{4}\cos 2E\right] - \frac{3\mathbf{E}(k)}{\pi}\mathcal{I}(\theta - E) - 3\mathcal{I}H\right\}\mathfrak{T} +$$
$$+ \frac{\eta}{\varkappa^2}\left[\mathcal{F}_1\left(E + \frac{\pi}{2}, e\right) - \mathbf{K}\left(e\right)\left(1 + \frac{2}{\pi}M\right) - \frac{1}{e^2}\ln\frac{e\sin E + \sqrt{1 - e^2\cos^2 E}}{\eta}\right]\mathfrak{N},$$

T.N. Sannikova. Displacement Norm in the Presence of an Inverse-Square Perturbing Acceleration in the Reference Frame Associated with the Velocity Vector // Astron. Rep. — 2024 — *Submitted for consideration for publication.*

The change-of-variable functions in O_2

θ is the true anomaly,

$$\vartheta = \sqrt{1+e^2+2e\cos\theta} = (1+e)\sqrt{1-k^2\sin^2\left(\frac{\theta}{2}\right)} = \eta\sqrt{\frac{1+e\cos E}{1-e\cos E}}\,,$$

$$\mathcal{I}(\theta - E) = -\frac{\beta(2+\beta^2)}{1+\beta^2}\left(\frac{e}{2}+\cos E\right) + \frac{2}{1+\beta^2}\sum_{n=2}^{\infty}\frac{n+1-(n-1)\beta^2}{n^2(n^2-1)}\beta^n\cos nE,$$

$$\eta = \sqrt{1-e^2}, \qquad \beta = \frac{e}{(1+\eta)}, \qquad k = \frac{2\sqrt{e}}{(1+e)},$$

$$\mathcal{I}H = \sum_{n=1}^{\infty}\frac{C_n}{n}\cos nM\,, \qquad C_n = \sum_{m=1}^{\infty}(-1)^m B_m(k) S_n^{0m}(e) k^{2m}$$

$S_n^{0m}(e) = X_n^{0m}(e) - X_{-n}^{0m}(e)$, $\quad X_k^{nm}$ are the Hansen coefficients

$$B_m(k) = \frac{1}{m}\sum_{s=0}^{\infty}\frac{(s+1)\cdots(s+m)}{(s+m+1)\cdots(s+2m)}\left[\frac{(2s+2m-1)!!}{(2s+2m)!!}\right]^2\frac{k^{2s}}{2s+2m-1}$$

Complete and incomplete elliptic integrals in Legendre form are

$$\mathbf{K}(k) = \int_0^{\pi/2}\frac{dx}{h(x,k)}\,, \quad \mathbf{E}(k) = \int_0^{\pi/2} h(x,k)\,dx, \quad \mathbf{D}(k) = \int_0^{\pi/2}\frac{\sin^2 x\,dx}{h(x,k)} = \frac{\mathbf{K}(k)-\mathbf{E}(k)}{k^2}\,,$$

$$\mathcal{F}_1(\varphi,k) = \int_0^{\varphi}\frac{dx}{h(x,k)}\,, \qquad \mathcal{F}_2(\varphi,k) = \int_0^{\varphi} h(x,k)\,dx,$$

$$\mathcal{F}_3(\varphi,k) = \int_0^{\varphi}\frac{\sin^2 x\,dx}{h(x,k)} = \frac{\mathcal{F}_1(\varphi,k)-\mathcal{F}_2(\varphi,k)}{k^2}\,,$$

where

$$h(x,k) = \sqrt{1-k^2\sin^2 x}.$$

The change-of-variable functions in O_2

We will express the increments of the elements $\delta\epsilon_n$ through the eccentric anomaly E and represent them in series, since the original expressions are complex functions of the eccentricity e:

$$\delta a = \frac{a\mathfrak{T}}{\varkappa^2(1-e^2)^2}\Bigg[\left(6e - 5e^3 - \frac{25e^5}{32} - \frac{29e^7}{256} - \frac{349e^9}{8192} - \frac{43e^{11}}{2048}\right)\sin E + \left(\frac{5e^2}{4} - \frac{13e^4}{16} - \frac{113e^6}{512} - \frac{179e^8}{2048} - \frac{2845e^{10}}{65536}\right)\sin 2E +$$
$$+\left(\frac{e^3}{2} - \frac{7e^5}{32} - \frac{27e^7}{256} - \frac{115e^9}{2048} - \frac{1085e^{11}}{32768}\right)\sin 3E + \left(\frac{27e^4}{128} - \frac{21e^6}{512} - \frac{347e^8}{8192} - \frac{995e^{10}}{32768}\right)\sin 4E +$$
$$+\left(\frac{3e^5}{32} + \frac{e^7}{256} - \frac{25e^9}{2048} - \frac{875e^{11}}{65536}\right)\sin 5E + \left(\frac{65e^6}{1536} + \frac{25e^8}{2048} + \frac{5e^{10}}{131072}\right)\sin 6E + \left(\frac{5e^7}{256} + \frac{85e^9}{8192} + \frac{245e^{11}}{65536}\right)\sin 7E +$$
$$+\left(\frac{595e^8}{65536} + \frac{1855e^{10}}{262144}\right)\sin 8E + \left(\frac{35e^9}{8192} + \frac{287e^{11}}{65536}\right)\sin 9E + \frac{1323e^{10}}{655360}\sin 10E + \frac{63e^{11}}{65536}\sin 11E\Bigg]$$

$\delta\epsilon_n$ are Fourier series of the form $\sum_{k=1}^{\infty} a_k(e)\sin kE$ or $\sum_{k=0}^{\infty} a_k(e)\cos kE$, where $a_k(e)$ are Maclaurin series in degrees of eccentricity with rational coefficients, and the first term of the series $a_k(e)$ has order $k-2$ or more. Therefore, when preserving terms up to a certain degree of eccentricity, a finite number of terms remain in the Fourier series.

Some of the expressions $\delta\epsilon_n$ have singularities at $e = 0$ or $e = 1$. But since averaging over the mean anomaly implies the ellipticity of the osculating orbit, that is, $0 < e < 1$, then singularity is not encountered in the calculations.

Norm of difference between osculating and mean elements in O_2

$$(d\mathbf{r})^2 = \delta r^2 + r^2(\delta u + \cos i\delta\Omega)^2 + r^2(\sin u\delta i - \sin i \cos u\delta\Omega)^2$$

$$(d\mathbf{r})^2 = \frac{\mathfrak{T}^2 a^2}{\varkappa^4}\sum_{k=0}^{\infty} a_{1k}(e)\cos kE + \frac{\mathfrak{N}^2 a^2}{\varkappa^4}\sum_{k=0}^{\infty} a_{2k}(e)\cos kE + \frac{W^2 a^2}{\varkappa^4}\sum_{k=0}^{\infty} a_{3k}(e)\cos kE + \frac{\mathfrak{T}\mathfrak{N} a^2}{\varkappa^4}\sum_{k=1}^{\infty} a_{4k}(e)\sin kE,$$

where the Maclaurin series for the coefficients $a_{nk}(e)$ always start with a term of order e^k.

Norm of difference between osculating and mean elements::

$$\varrho^2 = \|d\mathbf{r}\|^2 = \frac{\mathfrak{T}^2 a^2}{\varkappa^4}\sum_{k=0}^{\infty} a_{1k}(e)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\cos kE\,dE + \frac{\mathfrak{N}^2 a^2}{\varkappa^4}\sum_{k=0}^{\infty} a_{2k}(e)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\cos kE\,dE + \frac{W^2 a^2}{\varkappa^4}\sum_{k=0}^{\infty} a_{3k}(e)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\cos kE\,dE + \frac{\mathfrak{T}\mathfrak{N} a^2}{\varkappa^4}\sum_{k=1}^{\infty} a_{4k}(e)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\sin kE\,dE.$$

The last term is an odd function of E and disappears as a result of integration. In other cases, when integrating, we take into account the following integrals:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\,dE = 1,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\cos E\,dE = -\frac{e}{2},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r}{a}\cos kE\,dE = 0 \text{ при } k \geq 2.$$

Norm of difference between osculating and mean elements in O_2

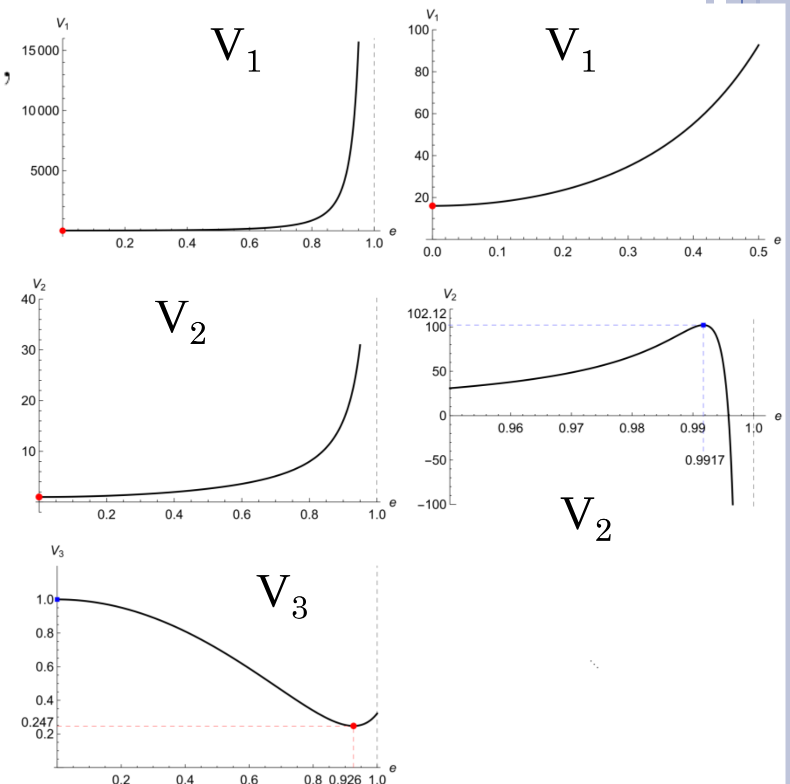
$$\varrho^2 = \|d\mathbf{r}\|^2 = \frac{\mathfrak{T}^2 a^2}{\varkappa^4}\left(a_{10}(e) - \frac{e}{2}a_{11}(e)\right) + \frac{\mathfrak{N}^2 a^2}{\varkappa^4}\left(a_{20}(e) - \frac{e}{2}a_{21}(e)\right) +$$

$$+\frac{W^2 a^2}{\varkappa^4}\left(a_{30}(e) - \frac{e}{2}a_{31}(e)\right) = \frac{a^2}{\varkappa^4}(V_1\mathfrak{T}^2 + V_2\mathfrak{N}^2 + V_3 W^2),$$

$$V_1 = \frac{1}{(1-e^2)^2}\left(16 + \frac{1121e^2}{8} + \frac{10793e^4}{512} - \frac{239033e^6}{18432} - \frac{17713751e^8}{18874368} - \frac{16112611e^{10}}{37748736} - \right.$$
$$- \frac{222199537e^{12}}{1006632960} - \frac{15794064133e^{14}}{123312537600} - \frac{78324446438359e^{16}}{969769255698432} - \frac{945456502286119e^{18}}{17455846602571776} -$$
$$\left. - \frac{135640061196163177e^{20}}{35624176739942400000} - \frac{2394973305173593870 7e^{22}}{8621050771066060800 00} - \frac{256113538574688068647e^{24}}{122610499855161753600 00} + \dots\right),$$

$$V_2 = \frac{1}{(1-e^2)^2}\left(1 + \frac{29e^2}{8} - \frac{2221e^4}{288} + \frac{1907e^6}{512} - \frac{265501e^8}{491520} - \frac{5955551e^{10}}{58982400} - \frac{108054281e^{12}}{3853516800} - \right.$$
$$- \frac{347992909e^{14}}{46242201600} - \frac{80402703347e^{16}}{17046725197824 0} + \frac{961519885723e^{18}}{454579338608640} + \frac{157193618149693e^{20}}{52384857115852800} +$$
$$\left. + \frac{12031971779882773e^{22}}{3771709712341401600} + \frac{700133658333747323 9e^{24}}{226637845826025553920 0} + \dots\right),$$

$$V_3 = \left(1 - \frac{39e^2}{32} + \frac{101e^4}{576} + \frac{599e^6}{6144} + \frac{19889e^8}{307200} + \frac{86891e^{10}}{1843200} + \frac{145911e^{12}}{4014080} + \frac{14979701e^{14}}{513802240} + \right.$$
$$+ \frac{286187473e^{16}}{11890851840} + \frac{402547717e^{18}}{19818086400} + \frac{3098641663e^{20}}{177628774400} + \frac{233207333021e^{22}}{15347126108160} +$$
$$\left. + \frac{300747020621e^{24}}{22455968071680} + \dots\right).$$



Behavior of functions V_1, V_2 and V_3 on the interval $e \in [0, 1)$:

- $\min(V_1) = 16$ for $e = 0$, $V_1 \to \infty$ for $e \to 1$, V_1 monotonically increases;
- $\min(V_2) = 1$ for $e = 0$, $\max(V_2) \approx 102.12175$ for $e \approx 0.991724$, $V_2 \to -\infty$ for $e \to 1$;
- $\min(V_3) \approx 0.247374$ for $e \approx 0.926173$, $\max(V_3) = 1$ for $e = 0$.

Function $V_2 < 0$ for $e > e_0 \approx 0.995862$. Thus $V_k > 0$ for $e < e_0$, therefore, on the interval $e \in [0, e_0]$ the norm ϱ^2 is positive and ϱ is a real number.
The dependence of V_1, V_2 and V_3 on e is shown in the figure.

Comparison of the difference norms in O_1 and O_2

$$\varrho_1^2 = \frac{a^2}{\varkappa^4}(A_1 S^2 + A_2 T^2 + A_3 W^2),$$

O_1

$$A_1 = \frac{1}{2}\left(2 + 3e^2\right),$$

$$A_2 = \frac{1}{(1-e^2)^2}\left(16 + \frac{3365e^2}{32} - \frac{12601e^4}{1152} - \frac{13327e^6}{2048} - \frac{226339e^8}{163840} - \mathcal{O}(e^{10})\right),$$

$$A_3 = 1 - \frac{39e^2}{32} + \frac{101e^4}{576} + \frac{599e^6}{6144} + \frac{19889e^8}{307200} + \mathcal{O}(e^{10}),$$

$$\varrho_2^2 = \|d\mathbf{r}\|^2 = \frac{a^2}{\varkappa^4}(B_1 \mathfrak{T}^2 + B_2 \mathfrak{N}^2 + B_3 W^2),$$

O_2

$$B_1 = \frac{1}{(1-e^2)^2}\left(16 + \frac{1121e^2}{8} + \frac{10793e^4}{512} - \frac{239033e^6}{18432} - \frac{17713751e^8}{18874368} - \mathcal{O}(e^{10})\right),$$

$$B_2 = \frac{1}{(1-e^2)^2}\left(1 + \frac{29e^2}{8} - \frac{2221e^4}{288} + \frac{1907e^6}{512} - \frac{265501e^8}{491520} - \mathcal{O}(e^{10})\right),$$

$$B_3 = 1 - \frac{39e^2}{32} + \frac{101e^4}{576} + \frac{599e^6}{6144} + \frac{19889e^8}{307200} + \mathcal{O}(e^{10}).$$

The formulas of the main result are identical up to the replacement of the components of the disturbing acceleration. The functions $A_n(e)$ and $B_n(e)$ are series in even powers of eccentricity. The functions $A_3(e)$ and $B_3(e)$ coincide, since the component W is the same for both reference frames. In the O_1 frame, the function $A_1(e)$ is a second-degree polynomial, whereas in the O_2 frame, $B_1(e)$ is an infinite series, $A_2(e)$ and $B_2(e)$ are series in both frames. Since at zero eccentricity the triangle $(-\mathfrak{N}, \mathfrak{T}, W)$ is identical to the triangle (S, T, W), then $A_1(0) = B_2(0)$, $A_2(0) = B_1(0)$ and $A_3(0) = B_3(0)$, i.e. the free terms coincide, as they should be.

Application
Initial data

T.N. Sannikova. Accounting for the Yarkovsky Effect in Reference Frames Associated with the Radius Vector and Velocity Vector // Astronomy Reports — 2022 — V.66, No.6 — p. 500-512

The article [Sannikova, 2022] considers model objects with different orbital eccentricities from 0 to 0.99 and other orbital and thermophysical characteristics, like the asteroid Bennu, and finds the mean orbital values of the components of the vector $\mathbf{P}$ in the $\mathcal{O}_1$ and $\mathcal{O}_2$ systems. Using the results of this article, we calculate the orbital shifts ϱ_1 and ϱ_2 for these model objects. The following constants were used in the calculations:

$1\,\mathrm{AU} = 1.495978707 \times 10^{11}\,\mathrm{m}$, $\varkappa^2 = 1.32712440041279419 \times 10^{20}\,\mathrm{m^3 s^{-2}}$, $1\,\mathrm{day} = 86400\,\mathrm{s}$.

For all occasions

$a = 1.126391025894812$ AU,

$S = 9.91079 \times 10^{-14}$ AU^3/day^2, $T = -5.10168 \times 10^{-14}$ AU^3/day^2, $W = 0$.

Results

e	$\mathfrak{T}$, 10^{-14} AU^3/day^2	$\mathfrak{N}$, 10^{-14} AU^3/day^2	ϱ_2, m	ϱ_1, m
0.001	-5.10168	-9.91079	129.185	129.185
0.01	-5.10155	-9.91054	129.245	129.231
0.10	-5.08887	-9.88585	135.127	133.848
0.20	-5.04976	-9.80969	152.479	147.865
0.30	-4.98212	-9.67805	180.585	171.674
0.40	-4.88179	-9.48280	219.968	206.987
0.50	-4.74156	-9.20998	273.527	258.152
0.60	-4.54897	-8.83547	348.406	335.067
0.70	-4.28099	-8.31451	461.304	461.827
0.80	-3.88832	-7.55138	658.382	711.424
0.90	-3.22864	-6.26976	1136.522	1448.588
0.99	-1.53792	-2.98595	5562.831	14545.945

The table contains the values of the tangential $\mathfrak{T}$ and normal $\mathfrak{N}$ components, the ϱ_1 and ϱ_2 displacements, calculated for different eccentricities e.

The table shows that with increasing e the magnitude of the periodic perturbations caused by the Yarkovsky effect increases, although the absolute values of the $\mathfrak{T}$ and $\mathfrak{N}$ components decrease. In the O_1 system, the S and T components do not depend on e, but the increase in ϱ_1 for large e is more pronounced than in the O_2 system. This may indicate an overestimation of short-period orbital perturbations for objects in highly elliptical orbits when calculated in the O_1 system.

In the general case, with small perturbing accelerations, characteristic of the Yarkovsky effect, the displacement of the osculating orbit relative to the average is small and can be neglected, taking into account only the secular drifts of the orbital elements.

Conclusion

- Expressions for the Euclidean (root mean square for the mean anomaly) norm of the difference between the osculating and mean elements are presented in two orbital reference systems: O_1, associated with the radius vector, and O_2, associated with the velocity vector.
- Short-period perturbations of the orbits of model asteroids with different orbital eccentricities due to the Yarkovsky effect are estimated.
- In the general case, with small perturbing accelerations, characteristic of the Yarkovsky effect, the displacement of the osculating orbit relative to the mean orbit is small and can be neglected, taking into account only the secular drifts of the orbital elements.
- As the eccentricity of the orbit increases, the magnitude of short-periodic disturbances increases.

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Thank you for your attention!

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