Semi-analytical theories of motion for the study of the dynamical evolution of planetary systems

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Ural Federal University named after the first President of Russia B.N.Yeltsin

Dedicated to the memory of Teacher
Konstantin Kholshevnikov

Introduction

Study of the motion of major planets in the Solar System (and other planetary systems)

- Two interrelated directions.
  - The first is the representation of motion with the highest possible accuracy on a short time scale ($10 - 10^3$ years).
    - Numerical ephemerides (DE/LE, EPM, and INPOP).
    - Analytical ephemerides (VSOP).
  - The second is a qualitative description of the main properties of motion on cosmogonic time scales ($10^4 - 10^{10}$ years).
    - Numerical theories (Applegate et al., 1986; LONGSTOP Project; Varadi et al., 2003; ...).
    - Symplectic theories (Ito and Tanikawa, 2002; Laskar ...).
    - **Semi-analytical** theories (Laskar ...).

Introduction

Semi-Analytical Theories

- The Gauss method (Davydov and Molchanov, 1971; Vashkov'yak, 1981; . . .)
  - The Gauss averaging requires no expansions in powers of the eccentricity or inclination.
- R-rings and R-toroids (Kondratev and Kornoukhov, 2018; 2019; 2020; 2021; 2022; 2023; Kondratev et al., 2021).
  - Triple averaging of the motion of a material point and is reduced to a chain of transformations: 1D Gaussian ring – 2D R-ring – 3D R-toroid.

Introduction

Semi-Analytical Theories

- The Halphen–Goryachev method (Sukhotin, 1981, 1984; Sukhotin and Kholshevnikov, 1986)
  - Doubly averaging the perturbing acceleration components over two fast variables is made.
  - Its computed the mutual secular perturbations of the outer planets over 800 kyr and of all nine planets over 200 kyr.
- Numerical integration of the secular system (Laskar, 1984, 1985, 1986, 1988).
  - The real motion of the Solar system planets can be traced in an interval of no more than 100 Myr.
  - The equations were integrated for 10 Gyr into the past and for 15 Gyr into the future.

Long-Period Evolution of Two-Planetary Systems

- The KAM theory (U.Locatelli and A.Giorgilli, 2000)
  - The Hamiltonian of the approximate secular model for the Sun – Jupiter – Saturn system generates two invariant tori that surround the orbits with Jupiter's and Saturn's initial data.
  - The orbits of Jupiter and Saturn are stable on an infinite time scale.
- The planar variant of the two-body problem in the absence of mean-motion resonances (Henrard and Libert, 2005; Libert and Henrard, 2005; 2007).
  - The solution is presented on the basis of the series by degrees of eccentricities (order 12).

Konstantin Kholshevnikov's contribution to the study of the Two-Planet Problem

The semi-analytical theory of motion of solar-type two-planetary system

- The Jacobi coordinate system
- Independent variables are the quantities

  $\tilde{a} = (a - a^0)/a^0 \quad (a^0$ is a mean value of $a)$,

  e,

  $\tilde{I} = \sin(I/2)$,

  and the longitudes $\alpha = l + g + \Omega$, $\beta = g + \Omega$, $\gamma = \Omega$.
  - Longitudes and $\tilde{I} = \sin(I/2)$ improves the analytical characteristics of the Hamiltonian and allowing to take advantage of its D'Alembertian properties (Kholshevnikov and Tublina, 1998).

The semi-analytical theory of motion of two-planet system

- Small parameter μ is the ratio of the mass of the planet to the mass of the star.
- Masses: m_0 is the mass of the star, $\mu m_0 m_1$, $\mu m_0 m_2$ are the masses of the planets,
- For the Sun – Jupiter – Saturn: $\mu = 10^{-3}$, $m_1 \approx 1$, $m_2 \approx 1/3$.
- Hamiltonian $h = h_0 + \mu h_1$
- Unperturbed part of the Hamiltonian h_0,

  perturbed dimensionless part of the Hamiltonian h_2

$$h_0 = -\frac{Gm_0 m_1}{2a_1} - \frac{Gm_0 m_2}{2a_2}, \qquad h_1 = \frac{Gm_0}{a_0} h_2 .$$

The semi-analytical theory of motion of two-planet system

The Expansion of the Hamiltonian of the Two-Planetary Problem into a Poisson series in all elements

$$h_2 = \sum A_{kn} x^k \cos ny.$$

A_{kn} are numerical coefficients;
$x = \{x_1, \ldots, x_6\}$ are positional variables;
$y = \{y_1, \ldots, y_6\}$ are angular variables;
$k = \{k_1, \ldots, k_6\}$, $n = \{n_1, \ldots, n_6\}$ are multi-indexes;

- Estimation and direct calculation of coefficients A_{kn} (Kholshevnikov et al., 2001; 2002)
  - The calculation of multiple integrals of elementary functions.
  - Summation limits.
  - Number of coefficients.

The semi-analytical theory of motion of a two-planet system

Summation limits and the number of coefficients N of the Hamiltonian expansion depending on the order of the theory

Order	$\|\|k\|\|_{max}$	$\|n_{3i-2}\|_{max}$	N
2	6	13	4×10^4
3	11	25	4×10^6
4	16	37	8×10^7

(Kholshevnikov et al., 2002):

"An increase in the speed of computers by $10^2 - 10^4$ times to the middle of the 21$^{\mathrm{st}}$ century will permit the expansion coefficients to be obtained with $6 - 8$ significant digits for one year of processor time" [of one CPU for 50000 coefficients]

The semi-analytical theory of motion of a two-planet system

Hamiltonian expansion with the Poisson Series Processor PSP (Kuznetsov and Kholshevnikov, 2004)

- Rational version of the Poisson Series Processor PSP (Ivanova, 1997).
- The expansion of the Hamiltonian with numerical parameters to μ^3 for the Sun – Jupiter – Saturn-like system is constructed.
- The parameters (masses of planets, mean values of semi-major axes of orbits) are given by polynomial variables.
- The expansion of the Hamiltonian with symbolic parameters to μ^2 is constructed.

The semi-analytical theory of motion of a two-planet system

Construction of averaged equations of motion (Kuznetsov and Kholshevnikov, 2006)

- Non-canonical parameterization of Poisson brackets.
    - To perform the averaging operation, we apply the Hori – Deprit method (the Lie transform method).
    - The method relies on Poisson brackets, which allows to turn out from canonical elements.
    - To carry out the calculations, it is sufficient to express the Poisson brackets in the system of phase variables needed by the researcher (Boronenko, 1975; Bordovitsyna et al., 1991; Kholshevnikov and Greb, 2001; Bordovitsyna and Avdyushev, 2007).
- Averaging over fast variables.
    - Fast variables are the mean longitudes of the planets
- Construction of variable change functions.

The semi-analytical theory of motion of a two-planet system

Performing Lie transformations (Kuznetsov and Kholshevnikov, 2006)

- Rational version of Echeloned Poisson Series Processor EPSP (Ivanova, 2001).
- A second improved approximation is constructed (in the terminology of Krylov and Bogolyubov).
  - Constructed to μ^3:
    - Averaged Hamiltonian $H = H_0 + \mu H_1 + \mu^2 H_2 + \mu^3 H_3$,
    - Generating function $T = \mu T_1 + \mu^2 T_2 + \mu^3 T_3$,

H_1	H_2	H_3	T_1	T_2	T_3	
414	8 168	9 717	60 672	1 044 513	858 311	terms

    - The right-hand sides of the averaged equations of motion.
  - Functions of change of variables is constructed to μ^2.

The semi-analytical theory of motion of a two-planet system

Orbital evolution of the two-planet system Sun – Jupiter – Saturn on cosmogonic intervals (Kuznetsov and Kholshevnikov, 2006)

- Numerical integration of equations of motion in averaged elements over 10 Gyr.
  - The Runge–Kutta method of 11$^{\text{th}}$ order (Danilov, 2008) and the Everhart method of 15$^{\text{th}}$ order (Everhart, 1974).
  - Range of variation of the averaged eccentricities e and inclinations (with respect to the ecliptic plane) I of the orbits of Jupiter and Saturn.

Jupiter	Saturn
$0.0170 \leqslant e \leqslant 0.0511$	$0.0195 \leqslant e \leqslant 0.0780$
$1.27^\circ \leqslant I \leqslant 2.00^\circ$	$0.734^\circ \leqslant I \leqslant 2.53^\circ$

  - The evolution of the orbital elements is almost periodic type.
  - Eccentricity and inclination values are separated from zero.

The semi-analytical theory of motion of a two-planet system

The conservation of area integrals in averaging transformations (Kuznetsov, 2009)

- As proved (Poincaré, 1905; Charlier, 1966; Kholshevnikov, 1991) the area integral is conserved in the system defined by the Hamiltonian H.
- But the components σ_x and σ_y (unlike σ_z and the energy integral E) are not conserved in the system defined by the finite segment of the Poisson series expansion of the averaged Hamiltonian H.

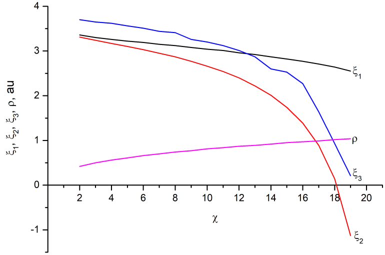
The semi-analytical theory of motion of a two-planet system

Stability of planetary systems with respect to masses (Kholshevnikov and Kuznetsov, 2010)

- Planetary masses m_k^0 were replaced by $m_k = \chi m_k^0$.
- According to Nacozy (1976), the Sun – Jupiter – Saturn system remains stable for 10^5 years up to $\chi = 29$.
- Sukhotin and Kholshevnikov (1986) used the Halphen–Goriachev method obtained that the Sun – Jupiter – Saturn system loses stability at $\chi \approx 99$.
- The motion described by the averaged system is much more stable than in the original system (Sukhotin and Kholshevnikov, 1986).

The semi-analytical theory of motion of a two-planet system

- Averaging with return to osculating elements (Kholshevnikov and Kuznetsov, 2010)



$$\xi_1 = \min[a_2(1 - e_2) - a_1(1 + e_1)]$$

(average elements),

$$\xi_2 = \min[a_2(1 - e_2) - a_1(1 + e_1)]$$

(osculating elements),

$$\xi_3 = \min|\mathbf{r}_2 - \mathbf{r}_1|,$$

ϱ is radius of Jupiter's sphere of action with respect to the Sun.

The semi-analytical theory of motion of a two-planet system

Stability of planetary systems with respect to masses (Kholshevnikov and Kuznetsov, 2010)

- The averaging with return to osculating elements gives a lower bound for the critical value of $\chi = 19$.
- A numerical integration up to 1 Gyr gives the critical value equal to $\chi = 22$ (Kholshevnikov and Kuznetsov, 2010).

- We can conclude that the averaging according to Gauss scheme gives us results far from the real ones.
- The averaging with return to osculating elements gives us results in a qualitative agreement with the real ones.

The semi-analytical theory of motion of a two-planet system

Resonances in planetary systems (Kuznetsov, 2010)

- Narrow and wide resonance zones (Sokolov, 1980; Sokolov, Kholshevnikov, 1981).
- The width of the resonance zone in the case of narrow resonance

$$\Delta a = \sqrt{\mu b}, \qquad b = \sum_{n\omega=0} \left| \frac{A_{kn} x^k}{1.5 \varkappa_2^2 a_2^{-5/2}} \right| .$$

A_{kn} are coefficients of the Poisson series representing the variable change function for the semi-major axis a_2.

Only the resonant terms are considered.

The semi-analytical theory of motion of a two-planet system

Resonances in planetary systems (Kuznetsov, 2010)

- For wide resonance, the size of the resonance zone is

$$\Delta a = \mu C + 2\sqrt{\mu b}, \qquad C = \sum_{n\omega \neq 0} \left| B_{kn} x^k \right| .$$

  Here B_{kn} are the coefficients of the echeloned Poisson series, representing the variable change function for the semi-major axis a_2.

  The non-resonant terms are treated as the sum of the moduli of the amplitudes.

- The coefficients of the variable change functions play a crucial role in analyzing the resonance properties of planetary systems.

The semi-analytical theory of motion of a two-planet system

- Despite the relative simplicity of the two-planet problem, its solution provided an opportunity to analyze a large number of related problems.
- In particular, the two-planet problem has served as a bridge from studies of the dynamical evolution of the Solar System to studies of the orbital evolution of exoplanet systems.

The semi-analytical theory of motion of a four-planet system

- The Jacobi coordinate system.
- Independent variables are the elements of the second Poincare system (canonical) — $L = \varkappa M\sqrt{a}$, eccentric elements ξ_1, ξ_2, oblique elements η_1, η_2 and mean longitude λ.
- Small parameter μ is the ratio of the sum of masses of the planets to the mass of the star.
- For the Sun – Jupiter – Saturn – Uranus – Neptune: $\mu = 10^{-3}$.
- Unperturbed part of the Hamiltonian h_0, perturbed dimensionless part of the Hamiltonian h_2, where h_0 and h_2 are written similarly as h_0 and h_2 for two-planet problem.

The semi-analytical theory of motion of a four-planet system

Hamiltonian expansion with the computer algebra system (CAS) Piranha (developed by F. Biscani)

- CAS Piranha is the echeloned Poisson series processor.
- The expansion of the Hamiltonian is constructed up to μ^3.
- All variables are saved in the series expansion as symbolic. The numerical coefficients are saved as the rational numbers with arbitrary precision (Perminov and Kuznetsov, 2015).

n	p_{max}	d_{max}	N_{terms}
1	6	60	36 853 938
2	4	20	6 017 416
3	3	15	6 314 479

n is the degree of μ, p_{max} is the maximum power of eccentric and oblique Poincare elements,

d_{max} is the maximum order of Legendre polynomials.

The estimation accuracy of the Hamiltonian expansion is $\sim 10^{-12}$ (Perminov and Kuznetsov, 2020).

The semi-analytical theory of motion of a four-planet system

Performing Lie transformations

- CAS Piranha is used for the impementation of the Hori–Deprit method.
- A second approximaton (Perminov and Kuznetsov, 2016) and a third approximation (2020) are constructed.
  - Constructed to μ^3:
    - Averaged Hamiltonian $H = H_0 + \mu H_1 + \mu^2 H_2 + \mu^3 H_3$.
    - Generating function $T = \mu T_1 + \mu^2 T_2$.

	H_1	H_2	H_3	T_1	T_2
p_{max}	6	4	2	6	2
N_{terms}	77 683	13 215 122	3 041 206 698	36 776 255	2 926 631 639

    - The right-hand sides of the averaged equations of motion.
  - Functions of change of variables is constructed to μ^2.

The semi-analytical theory of motion of a four-planet system

Orbital evolution of the four-planet system Sun – Jupiter – Saturn – Uranus – Neptune (Perminov and Kuznetsov 2018, 2020)

- Numerical integration of equations of motion in averaged elements on cosmogonic time intervals.
- The Everhart method of 15th order (Everhart, 1974) on time interval 100 Myr and Gragg-Bulirsch-Stoer method of 12th order (Avdyushev, 2015) on time interval 10 Gyr.
- The limits of variation and the periods of the change of the averaged orbital elements of the giant planets of the Solar system are calculated and compared with other theories.
- The evolution of the orbital elements is almost periodic type.
- Eccentricity and inclination values are separated from zero.

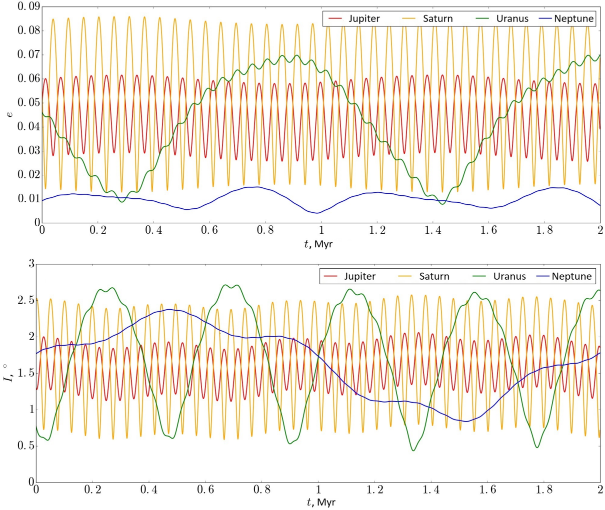
The semi-analytical theory of motion of a four-planet system

	e_{min}	e_{max}	I_{min}, °	I_{max}, °	e_{min}	e_{max}	I_{min}, °	I_{max}, °
	Jupiter				Saturn			
SA3	0.0208	0.0658	1.0922	2.0648	0.0099	0.0898	0.5532	2.6015
CS	0.0217	0.0651	1.0932	2.0616	0.0094	0.0875	0.5630	2.5978
WH	0.0251	0.0619	1.0936	2.0626	0.0093	0.0875	0.5616	2.5951
	Uranus				Neptune			
SA3	0.0032	0.0750	0.3894	2.7676	0.0016	0.0169	0.7756	2.3769
CS	0.0025	0.0745	0.3756	2.7763	0.0014	0.0171	0.7748	2.3733
WH	0.0034	0.0737	0.3768	2.7751	0.0024	0.0161	0.7742	2.3733

	Jupiter	Saturn	Uranus	Neptune	Jupiter	Saturn	Uranus	Neptune
	periods of orbital eccentricities, years				periods of orbital inclinations, years			
SA2	54 290	54 290	1 129 803	538 101, 364 896	49 213	49 213	432 965	1 876 305
SA3	54 688	54 688	1 136 388	537 416, 364 880	49 215	49 215	432 913	1 875 638
CS	54 693	54 693	1 133 548	534 230, 363 076	49 102	49 102	432 552	1 873 823
WH	54 747	54 747	1 134 679	534 765, 363 438	49 151	49 151	432 984	1 875 689
LA	54 060	54 060	1 117 240	535 100, 361 810	49 220	49 220	431 280	1 872 830

SA2 – the second order of the semi-analytical theory (Perminov and Kuznetsov, 2018),
SA3 – the third order of the semi-analytical theory (Perminov and Kuznetsov, 2020),
CS – Cowell–Stormer method from the program NBI (Goldstein, 1996; Varadi, 1999),
WH – Wisdom–Holman method from Rebound (Rein and Tamayo, 2015),
LA – Laskar's secular theory (Laskar, 1990).

The semi-analytical theory of motion of a four-planet system



The evolution of averaged orbital eccentricities (top figure) and inclinations (bottom figure) of giant planets of the Solar system over time interval 2 Myr in the third order approximation of the semi-analytical motion theory.

The semi-analytical theory of motion of a four-planet system

The accuracy of the constructed semi-analytical four-planet motion theory

- The discrepancies between periods of the change of the orbital inclinations obtained by numerical methods and semi-analytical motion theory do not exceed 0.1% for all planets.
- These same discrepancies for the orbital eccentricities of Jupiter and Saturn are about 0.01%.
- For Uranus and Neptune, these discrepancies for the orbital eccentricities are not exceeded 0.3% and 0.6% correspondingly.
- The discrepancies of the orbital eccentricities reach 1% in comparison with the results of Laskar's work.

The orbital evolution of extrasolar planetary systems

Constructed theory of motion of the second order is used for the investigation of the orbital evolution of extrasolar planetary systems with moderate values of orbital eccentricities and inclinations.

- Some orbital elements of extrasolar planetary systems are known with some uncertainties or undetermined, due to the specific limitations of the observing methods.
- The orbital evolution of extrasolar planetary systems are investigated by numerical integration of analitically constructed equations of motion for various initial conditions.
- The ranges of variation of the orbital elements can be determined as a function of the initial conditions.
- The assumption that the observed planetary systems are stable can be used to exclude initial conditions leading to extreme growth in the orbital eccentricities and inclinations.

HD 39194, HD 141399, HD 160691 (Perminov, Kuznetsov, 2019)

- All planets in these systems were detected via Doppler spectroscopy.
- HD 39194 has 3 super-Earths located within ≈ 0.24 AU around the host star. Arguments of all periapsis are known.
- HD 141399 – 4 Jupiter-like planets with $a \in [0.415, 5]$ AU.
- HD 160691 – 1 mini-Neptune and 3 Jupiter-like planets with $a \in [0.09, 5.235]$ AU.
- Nominal values of eccentricities for all orbits are not exceed 0.26.
- Values of all orbital inclinations and ascending nodes are unknown from observations.

The orbital evolution of extrasolar planetary systems

HD 39194, HD 141399, HD 160691 (Perminov, Kuznetsov, 2019)

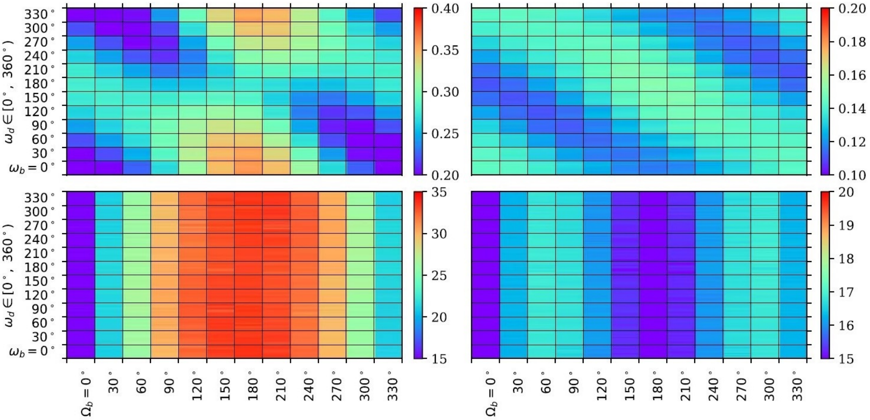
- The reference plane should be selected (for example, it can be the orbital plane of the most massive planet in the system). Initial values of I and Ω set to 0° for this planet.
- The orbital inclinations of the remaining planets vary.
- Different spatial configurations of the planets in their orbits were achieved by varying both Ω and ω.
- Estimates of the theoretical radii of convergence for orbital eccentricities R_e and inclinations R_I should be calculated for the series representing the equations of motion.
- If $e \leq R_e$ and $I \leq R_I$, this guarantees the convergence of the series representing the equations of motion, and the suitability of the theory of motion under the specified conditions.

The orbital evolution of extrasolar planetary systems

GJ 3138 (Perminov, Kuznetsov, 2022)

- Two super-Earths and one sub-Neptune orbiting around GJ 3138 were discovered by Doppler spectroscopy.
- Planetary system is compact – all semimajor axes < 0.7 AU.
- Nominal values of orbital eccentricities of three planets are 0.19 (GJ 3138 c), 0.11 (b) and 0.32 (d).
- Values of I, Ω and ω are unknown from observations and vary for different initial conditions of modelling.
- The reference plane is the orbital plane of the outermost planet GJ 3138 d (the most massive planet).
- The numerical integration of the equation of motion in averaged elements is performed by Gragg–Bulirsch–Stoer method of 7th order (Press et al. 2007) over 1 Myr.

The orbital evolution of extrasolar planetary systems



An example of integration results. Maximum values of averaged orbital eccentricities of planets GJ 3138 c (top left) and b (top right), averaged orbital inclinations of planets GJ 3138 c (bottom left) and b (bottom right) for nominal initial values of orbital eccentricities and initial inclinations of planets c and b are 15°.

The orbital evolution of extrasolar planetary systems

GJ 3138 (Perminov, Kuznetsov, 2022)

- The results of semi-analytical theory of motion are compared with direct numerical simulation, which is performed by Wisdom–Holman method (Rein, Tamayo, 2015).
- Chaotic properties are studied using MEGNO indicator.
- The radii R_e and R_I are calculated.
- If $I \geq 20^\circ$, the initial conditions leading to Lidov–Kozai resonance appears.
- The maximum increase in the averaged orbital eccentricity:
  GJ 3138 c – the condition $(\Omega_b - \Omega_c) + (\omega_b - \omega_c) = 180^\circ$.
  GJ 3138 b – the condition $(\Omega_b - \Omega_c) + (\omega_b - \omega_c) = 0^\circ$.
- The most probable values of the orbital eccentricities and inclinations from the point of view of stability are identified.

The orbital evolution of extrasolar planetary systems

Kepler-51 (Perminov, Kuznetsov, 2024 (in print))

- Kepler-51 has 3 super-Earths located within ≈ 0.9 AU around the host star. The values of orbital eccentricities are not exceed ≈ 0.06. All planets were discovered by transit method and orbital inclinations are not exceed 1°.
- The numerical integration of the equation of motion in averaged elements is performed by Gragg–Bulirsch–Stoer method of 7th order and the Posidonius software package (Blanco-Cuaresma, Bolmont, 2017), taking into account tidal interaction over 100 Myr for various initial conditions.
- The study shows that the compact planetary system Kepler-51 is not resonant and its orbital evolution is stable over time interval of 100 Myr.

The semi-analytical theory of motion of 8-planet system

- According to the *exoplanet.eu* database, 24 five-planet systems, 11 six-planet systems, 5 seven-planet systems and 1 eight-planet extrasolar system Kepler-90 have been discovered to date.
- At present, the authors are constructing an eight-planet semi-analytical theory of motion.
- Jacobi coordinate system and the second system of Poincare elements are used.
- The osculating Hamiltonian is constructed up to μ^3.
- The averaged Hamiltonian and the equation of motion in averaged elements will be construct up to μ^3.

The resonant semi-analytical theory of motion

- Jupiter and Saturn are close to 2:5 mean motion resonance; $2\nu_J - 2\nu_S \approx -1 \cdot 10^{-5}$.
- Uranus and Neptune are close to 1:2 mean motion resonance; $\nu_U - 2\nu_N \approx -4 \cdot 10^{-6}$.
- Neptune and Pluto – mean motion resonance 2:3; $2\nu_N - 3\nu_P \approx 6 \cdot 10^{-7}$.
- Many extrasolar systems with planets in the vicinity of MMR.
- At present, the authors are constructing a resonant four-planet semi-analytical theory of motion.

The resonant semi-analytical theory of motion

Canonical change of variables

- Jacobi coordinate system and the second system of Poincare elements are used.
- In the case of mean motion resonance $k_n : k_m$ between planets n and m the canonical change of variables is used

$$L_n^{'} = \frac{1}{k_n} L_n, \qquad \lambda_{nm}^{'} = k_n \lambda_n - k_m \lambda_m,$$

$$L_m^{'} = \frac{k_m}{k_n} L_n + L_m, \qquad \lambda_m^{'} = \lambda_m.$$

Here λ_{nm} is the critical argument of the resonance (this new variable is slow in the vicinity of the resonance).

Conclusions

- Productive ideas on the development of the semi-analytical theory of the N-planet problem laid down by **Professor Konstantin Kholshevnikov** continue to be realized in new versions of the theory, bringing new scientific results.

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Thank you for your attention!