Semi-analytical theories of motion for the study of the dynamical evolution of planetary systems

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Abstract. We present the results of construction and use of semi-analytical theories of planetary motion initiated by Professor Konstantin Kholshevnikov. Theories of motion for two- and four-planet systems, as well as their applications to the study of long-period evolution, stability, and occurrences of chaos in the Solar System and extrasolar planetary systems are discussed. We announce the development of an eight-planet theory of motion and a version of the theory to account for mean-motion resonances.

Introduction

From the 18th century until the mid-20th century, all the theories of planetary motion needed for practice were constructed analytically by the small parameter method. In the early 20th century, Lyapunov and Poincaré established the convergence of the corresponding series for a sufficiently small time interval. Subsequently, K. Kholshevnikov estimated this interval to be on the order of several tens of thousands of years, which is in agreement with numerical experiments. The first works describing analytically (in the first approximation) the evolution on cosmogonic time scales appeared in the first half of the 19th century (Laplace, Lagrange, Gauss, Poisson). The averaging method was developed in the early 20th century based on these works. In the first half of the 20th century, the averaging method introduced by Gauss as an approximate one became an exact one, at least formally (the series were handled as polynomials), through the works by H. Zeipel, N.M. Krylov, and N.N. Bogolyubov. In the 1960s, G. Hori and, independently, A. Deprit suggested a method of Lie transforms. Detailed reviews of the works on the orbital evolution of Solar System major planets see in [6].

Powerful analytical and numerical methods have made significant progress in describing the orbital evolution of planetary systems. In this work we present the results of construction and use of semi-analytical theories of planetary motion initiated by Professor Konstantin Kholshevnikov. Theories of motion for twoand four-planet systems, as well as their applications to the study of long-period evolution, stability, and occurrences of chaos in the Solar System and extrasolar planetary systems are discussed. We are also announcing the development of an eight-planet theory of motion and a version of the theory to account for mean-motion resonances.

1. Semi-analytical theory of motion of a two-planet system

The construction of the theory of planetary motion was carried out with the aim to study the evolution of solar-type planetary systems. We used the Jacobi coordinate system as the most suitable coordinate system for studying the evolution of planetary orbits [15]. The form of the Poisson expansion of the Hamiltonian in all elements was given in [4]. In [5], the expansion coefficients for the Hamiltonian of the two-planet Sun–Jupiter–Saturn problem were obtained using a simple algorithm reduced to the calculation of multiple integrals of elementary functions, the convergence domain was found, and the summation limits and the number of coefficients of the desired expansion were estimated. In [10], the expansions of the Hamiltonian of the two-planet problem into the Poisson series in all elements were constructed with the help of the PSP Poisson Series Processor [2].

We used the Hori–Deprit method to construct the averaged Hamiltonian of the two-planetary problem and the right-hand sides of the equations in average elements accurate to the third order of a small parameter, the generating function of the transform and the change of variables expressions to the second order of a small parameter [12]. Analytical transformations were performed with the help of the rational version of the echeloned Poisson series processor EPSP [3]. The evolution of the two-planet Sun–Jupiter–Saturn system was studied by numerically over 10 Gyr [11, 12].

The constructed theory was used to study the stability of planetary systems with respect to masses [7]. The study of Lagrange stability with respect to masses allows us to obtain upper limits for masses of extrasolar planets. In the Solar System, when the masses of Jupiter and Saturn increase by 20 times, these planets can have close approaches on a time scale of 1 Myr. Close approaches appear when analyzing osculating elements; they are absent in the mean elements. A similar situation takes place in the case of studied exoplanetary systems.

Our results established the bounds of applicability of the theorem that the area integral is conserved during averaging transformations: taking into account a finite number of terms in the series representing the averaged Hamiltonian, only one of the three components of the area vector is conserved, namely, the one corresponding to the longitudes measuring plane. We concluded that the non-conservation of the components σ_x and σ_y of the area integral is due to a failure to include small terms that are neglected when representing the averaged Hamiltonian in the form of a Poisson series.

We proposed the method for describing the resonance properties of planetary systems [8]. Our estimates of the resonance values of the semi-major axes and widths of resonance zones in relative units for characteristic values of the small parameter of the problem make it easy to classify and describe the resonance properties of planetary systems.

2. Enhancement of the semi-analytic theory of motion of the N-planet system

The development of the constructed semi-analytic theory became the theory of motion of four-planet systems [14]. The Hamiltonian expansion of the four-planetary problem into the Poisson series in elements of Poincaré second system is constructed up to third degree in the small parameter. The averaged Hamiltonian and the equations of motion in averaged Poincaré elements are constructed by the Hori–Deprit method up to third degree of the small parameter. The functions for the change of variables are obtained in second approximation and used for the transformation between osculating and averaged orbital elements. The transformations were performed analytically using the Piranha computer algebra system [1]. The constructed analytical equations of motion are applied to the study of the orbital evolution of the Solar System's giant planets on long time scales. The amplitudes and periods of the planetary motion are in good agreement with numerical theories. The investigation of the dynamical evolution of the chosen extrasolar planetary systems was performed in the framework of the theory of motion of the second order in planetary masses [13].

The next stage in the development of the semi-analytic theory of planetary motion is the construction of the eight-planet theory of motion and a version of the theory to account for mean-motion resonances.

Conclusion

Productive ideas on the development of the semi-analytical theory of the N-planet problem laid down by Professor Konstantin Kholshevnikov continue to be realised in new versions of the theory, bringing new scientific results.

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