Finite-point approximations of fields of attraction and their verification

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Analytical Methods of Celestial Mechanics
in memoriam of Konstantin Kholshevnikov
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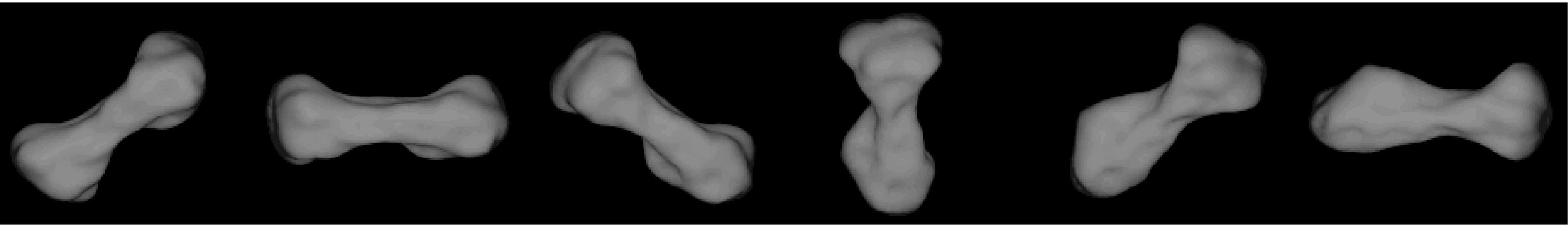
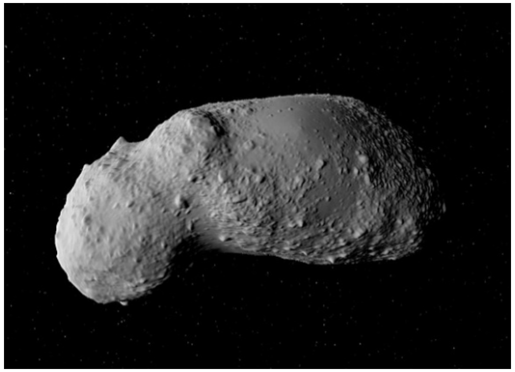
Euler International Mathematical Institute, St. Petersburg, RUSSIA

Complex shape of small celestial bodies



(216) Kleopatra.
Credit: Stephen Ostro et al. (JPL), Arecibo Radio Telescope, NSF, NASA

67P/Churyumov–Gerasimenko.
Credit: ESA

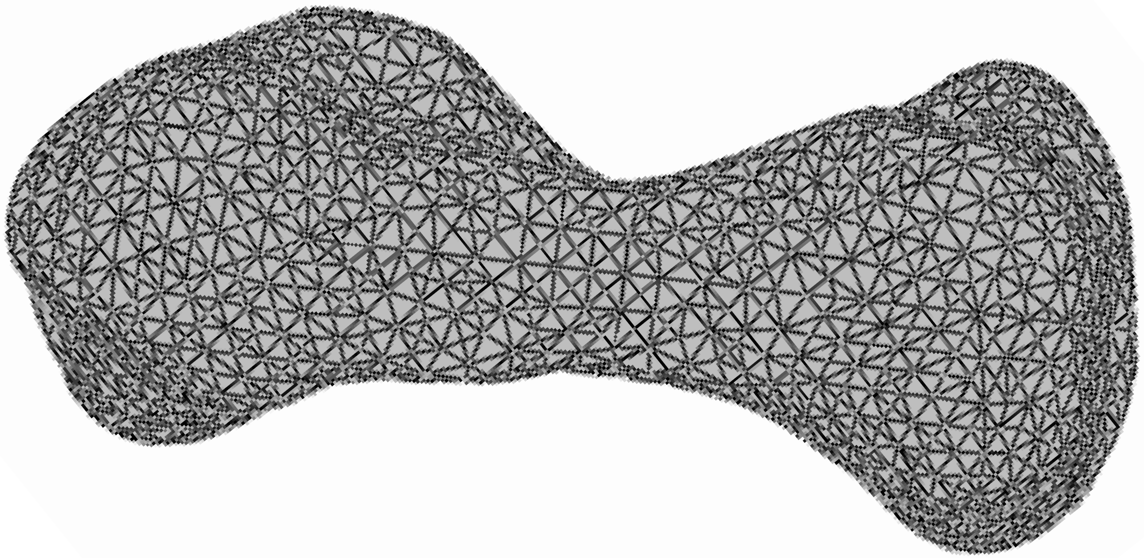


(25143) Itokawa.
Credit: JAXA

(433) Eros.
Credit: NASA

Three-dimensional models of celestial bodies

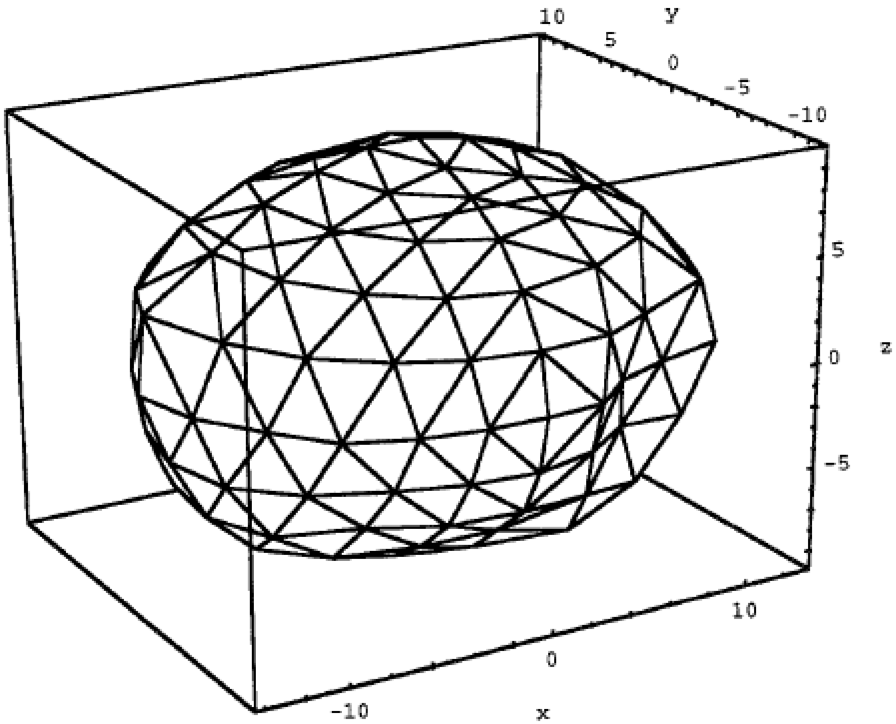
Triangulation meshes of various calibers are built based on the results of observations to approximate body's surface.



Asteroid (216) Kleopatra

Dataset of three-dimensional models for some celestial bodies:
https://sbn.psi.edu/pds/shape-models/

Gravitational potential of homogeneous polyhedron



Phobos' physical surface

Werner R.
The gravitational potential of a homogeneous polyhedron or don't cut corners //
Celestial Mechanics and Dynamical Astronomy. 1994. V. 59, no. 3. P.253 – 278

Werner R., Scheeres D.
Exterior gravitation of a polyhedron derived and compared with harmonic and mascon gravitation representations of asteroid 4769 Castalia //
Celestial Mechanics and Dynamical Astronomy. 1996. V. 65, no. 3. P. 313-344.

$$\frac{2}{|\vec{r}|} = div\left(\frac{\vec{r}}{|\vec{r}|}\right) \Rightarrow \iiint \frac{1}{|\vec{r}|} dV = \frac{1}{2} \iint \left(\vec{n}, \frac{\vec{r}}{|\vec{r}|}\right) dS$$

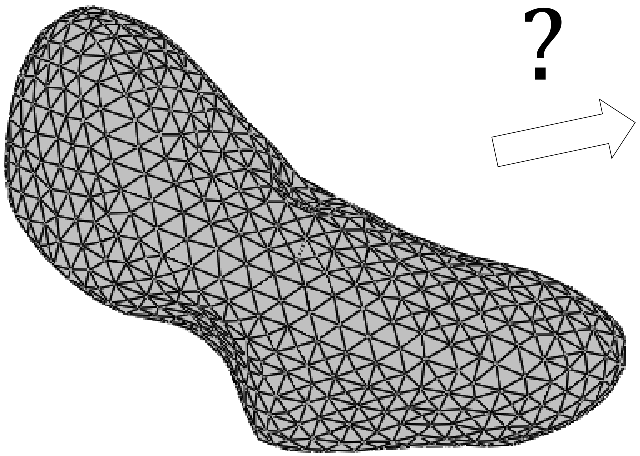
Werner–Scheeres formula

$$U = -\frac{1}{2} G\, \rho \left(\sum_{e \in E} \boldsymbol{r}_e\, \boldsymbol{E}_e \boldsymbol{r}_e\, L_e - \sum_{f \in F} \boldsymbol{r}_f\, \boldsymbol{F}_f \boldsymbol{r}_f\, \omega_f \right)$$

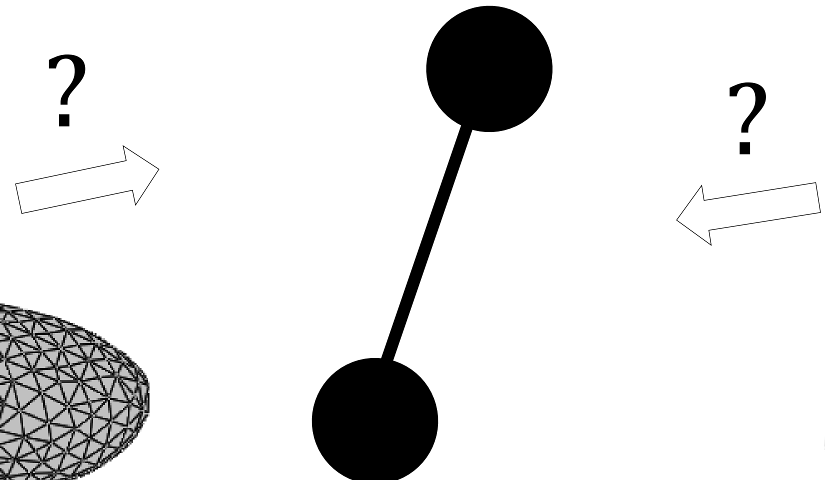
G – gravitational constant,
$\rho = const$ – density contrast,
$\boldsymbol{r}_e$– vector from test point to an arbitrary point of the edge e,
$\boldsymbol{r}_f$– vector from test point to an arbitrary point of the facet f,
$\boldsymbol{E}_e$ – matrix defined by the normals to the edge e lying in the incident facets with it, and the external normals to these facets,
$\boldsymbol{F}_f$ – matrix defined by the external normal to the facet f,
$L_e = L_e(\boldsymbol{r}),\ \omega_f = \omega_f(\boldsymbol{r})$.

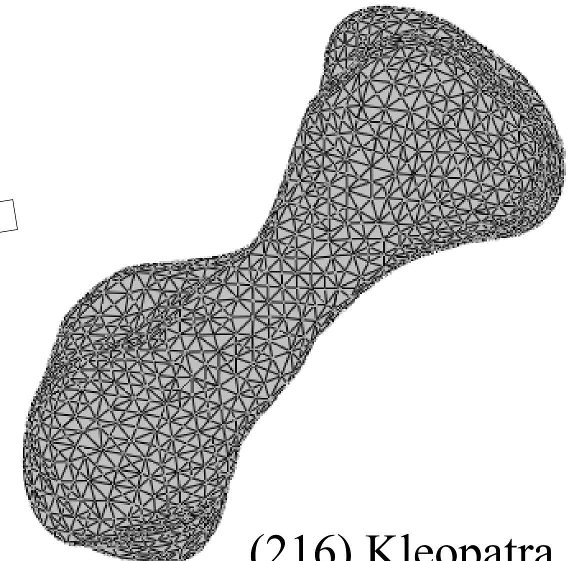
There are too many terms in the expression for the potential,
so there is no chance for an analytical study of the dynamics in bodies vicinity.

V.V. Beletsky's proposal: approximation of elongated asteroid by dumbbell-shaped body



(433) Eros





(216) Kleopatra

The question is
how can one find the dumbbell parameters?

[*] Beletsky V. V., Ponomareva O. N. Parametric analysis of the stability of relative equilibrium in a gravitational field // Cosmic Research. 1990. V. 28, no. 5. P. 573-582.
[**] Beletsky V. V., Rodnikov, A. V. Stability of triangle libration points in generalized restricted circular three-body problem // Cosmic Research. 2008. V. 46, no. 1. P. 40-48.

Cluster analysis. Steinhaus's idea

Let A be a set of points: $A = (x_1, x_2, \dots, x_N)$.
The aim is to partition N points into 2 sets

$$\boldsymbol{\mathcal{A}} = (A_1, A_2)$$

so as

$$A = A_1 \cup A_2,$$

$$A_1 \cap A_2 = 0,$$

$$\max_{\boldsymbol{\mathcal{A}}} \rho_{12},$$

where
$\rho_{12} = dist(C_1, C_2)$; C_i – centroid of A_i

Hugo Dyonizy Steinhaus

14.01.1887 Jasło (Galicja) (Austria-Hungary)
25.02.1972 Wrocław (Poland)

[*] Steinhaus H. Sur la division des corp materiels en parties // Bull. Acad. Polon. Sci. 1956. IV (Cl.III). P. 801–804.

K-means algorithm

Let A be a set of points

For a given an *initial* points $S_1^0, S_2^0 \in A$, the algorithm proceeds by alternating between two steps.

1. **Assignment step:**

   A point from A is assigned to A_i, if this point is closest to the point S_i^0, $i = 1..2$

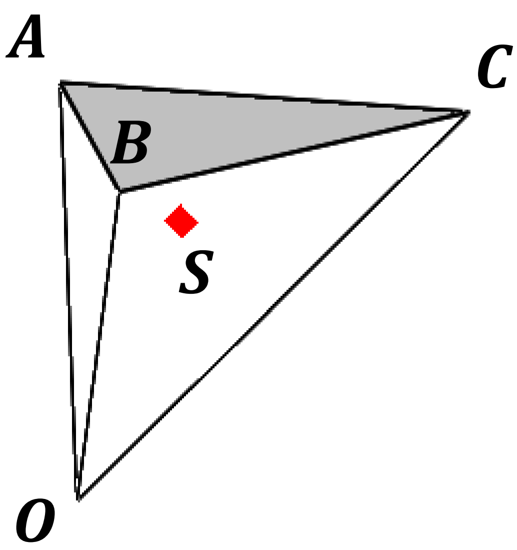
2. **Update step:**

   Calculate centroids S_1, S_2 of sets A_1, A_2.

   - If $\left|S_i S_i^0\right| < \varepsilon, \forall\, i = 1,2$, then the clustering is COMPLITE!
   - Otherwise, do the reassignment $S_1^0 = S_1,\ S_2^0 = S_2$, and REPEAT the loop from the first step.

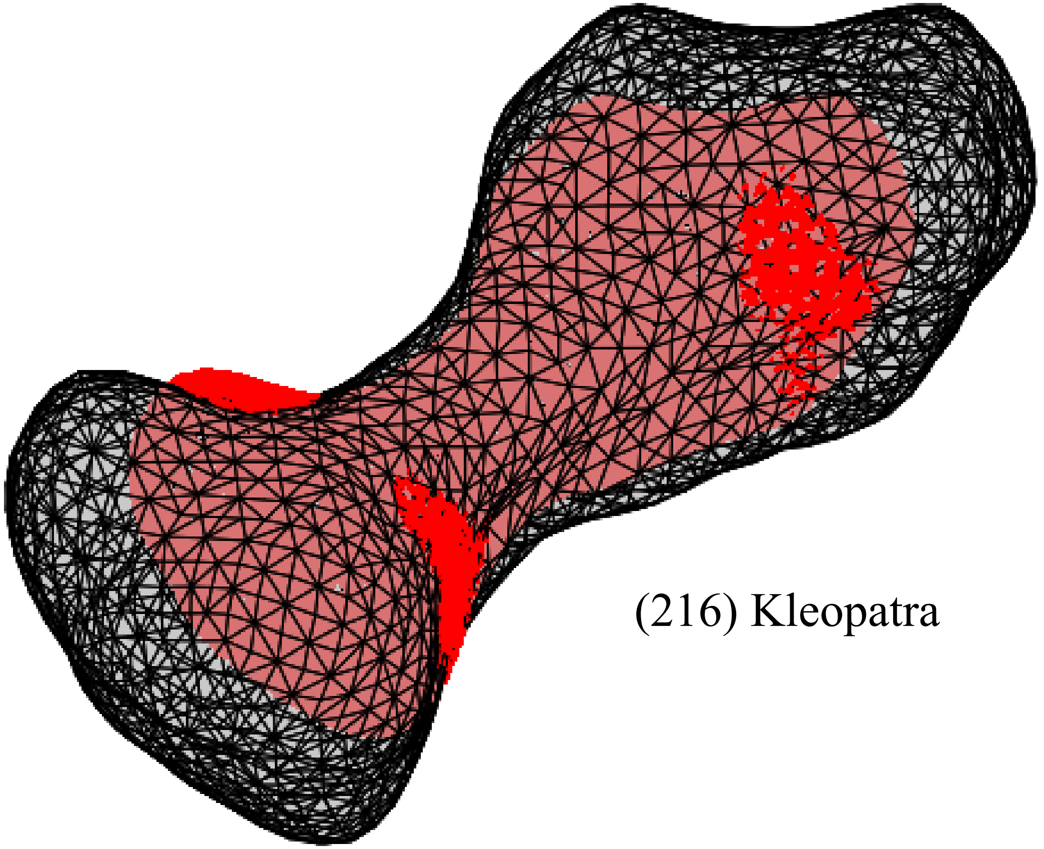
[*] Lloyd S.P. Least squares quantization in PCM // IEEE Transactions on Information Theory. 1982. V. 28, no. 2. P. 129–136.

Tetrahedral cells of triangulation and its centroids



$$\overrightarrow{\boldsymbol{OS}} = \frac{1}{4}\left(\overrightarrow{\boldsymbol{OA}} + \overrightarrow{\boldsymbol{OB}} + \overrightarrow{\boldsymbol{OC}}\right)$$

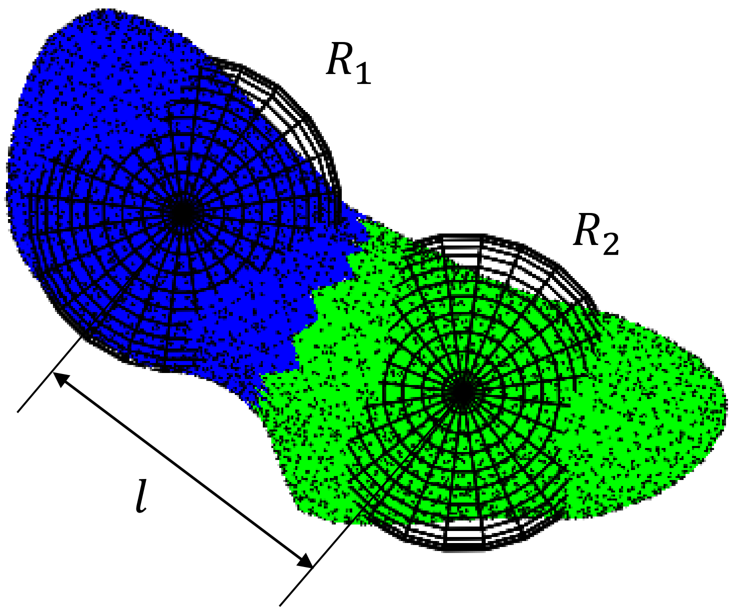
$$\mathrm{Vol}_{\boldsymbol{s}} = \frac{1}{6}\overrightarrow{\boldsymbol{OA}} \cdot \left(\overrightarrow{\boldsymbol{OB}} \times \overrightarrow{\boldsymbol{OC}}\right)$$



(216) Kleopatra

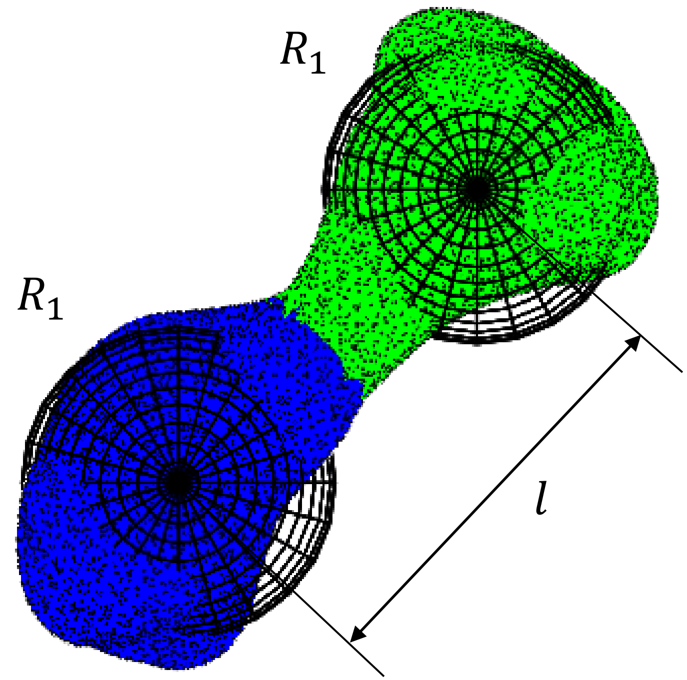
[*] Hitt D.L., Pearl J.M. Asteroid gravitational models using mascons derived from polyhedral sources // SPACE Conferences and Exposition, 13 - 16 September 2016, Long Beach, California

K-means clustering in representation celestial body mass distribution



(433) Eros

R_1 =6675.88 m, R_2 = 6674.2 m,
l = 13936 m



(216) Kleopatra

R_1= 43548.84 m, R_2 = 44249.16 m,
l = 117800 m

[*] Burov A.A., Nikonov V.I. et al. On the use of the K-means algorithm for determination of mass distributions in dumbbell-like celestial bodies // Rus. J. Nonlin. Dyn., 2018, V. 14, no. 1. P. 45-52

Augmented potential and libration points

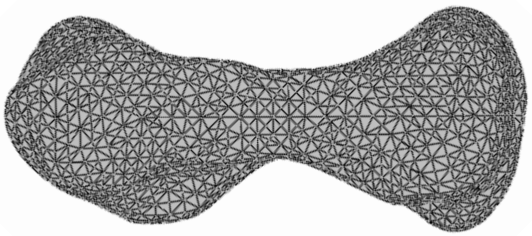
Libration points correspond to the critical points of the amended potential, which has the form

$$U_\omega = -\frac{1}{2}\omega^2(x^2 + y^2) + U$$

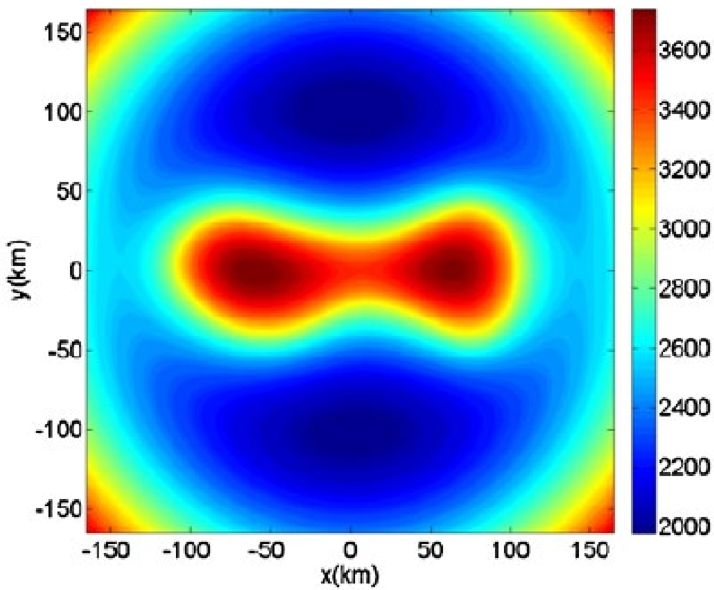
ω – angular velocity of rotation.
The stable rotation of the body is assumed to carry out about one of its inertia axes;
U – gravitational potential.

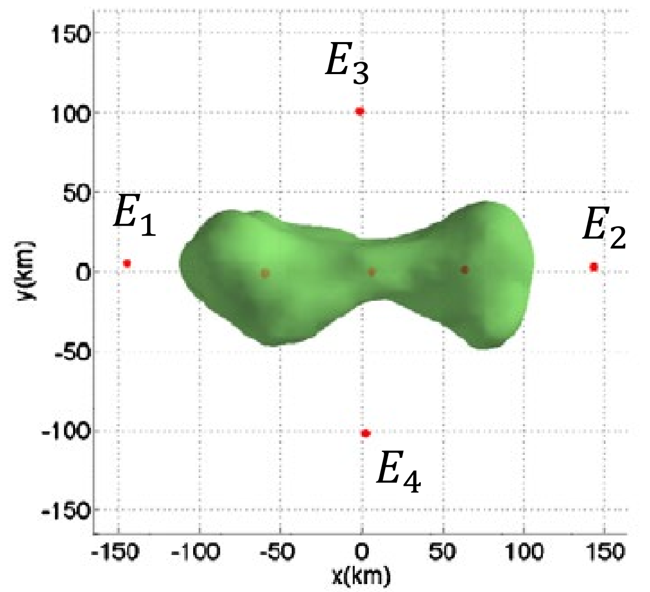
[*] Pravec P., Harris A. W., Michalowski T. Asteroid Rotations // In: Bottke, W.F., Cellino, A., Paolicchi, P., Binzel, R.P. (Eds.), Asteroids III. Univ. of Arizona Press, Tucson, 2002, p. 113–122.
[**] Kaasalainen M. Interpretation of lightcurves of precessing asteroids // Astronomy & Astrophysics. 2001. V. 376, no. 1. P. 302 - 309.
[***] Pravec P., et al. Tumbling asteroids // Icarus. 2005. V. 173. P. 108–131.

Determination of dumbbell parameters via libration points



(216) Kleopatra



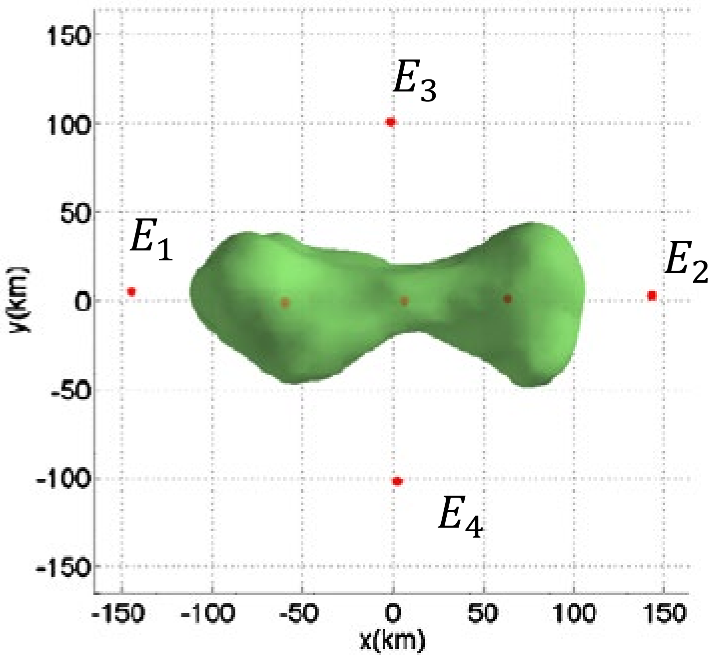


For asteroid (216) Kleopatra, the position of the axis of rotation and its angular velocity are known from observations. Also, using the Werner-Scheeres approach based on a triangulation mesh, an approximation of the field of attraction was constructed in [*]. It makes possible to determine the libration points [*]:

Libration points [*]	x (km)	y (km)	z (km)
E1	142.852	2.44129	1.18154
E2	−1.16383	100.740	−0.545312
E3	−144.684	5.18829	−0.272463
E4	2.22985	−102.102	0.271694

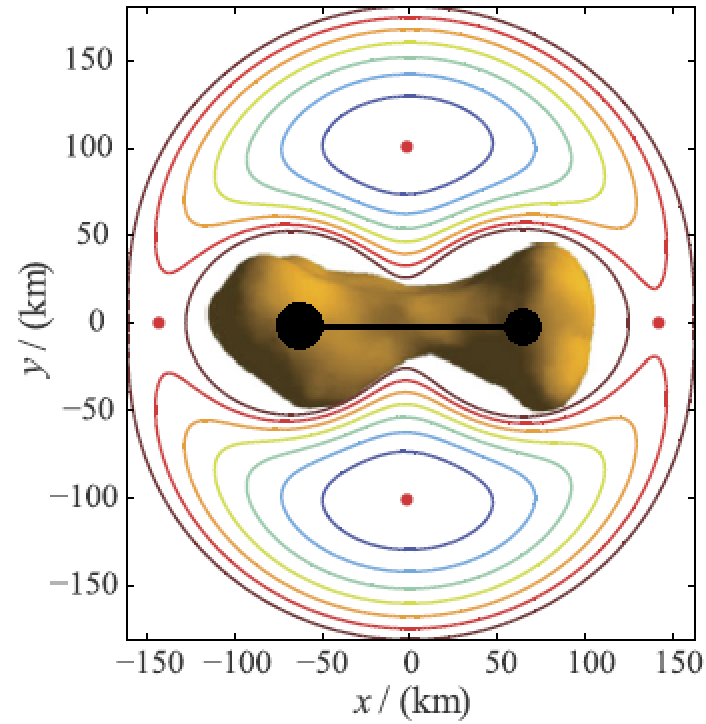
[*] Wang, X.Y., Jiang, Y., Gong, S.P. Analysis of the potential field and equilibrium points of irregular-shaped minor celestial bodies // Astrophys. Space Sci. 2014. V. 353, no. 1. P. 105–121.

Determination of dumbbell parameters via libration points



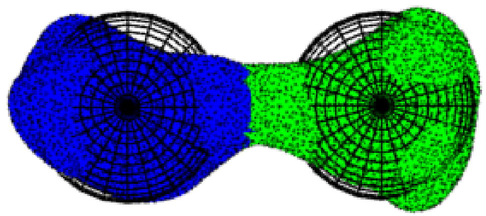
From [*]

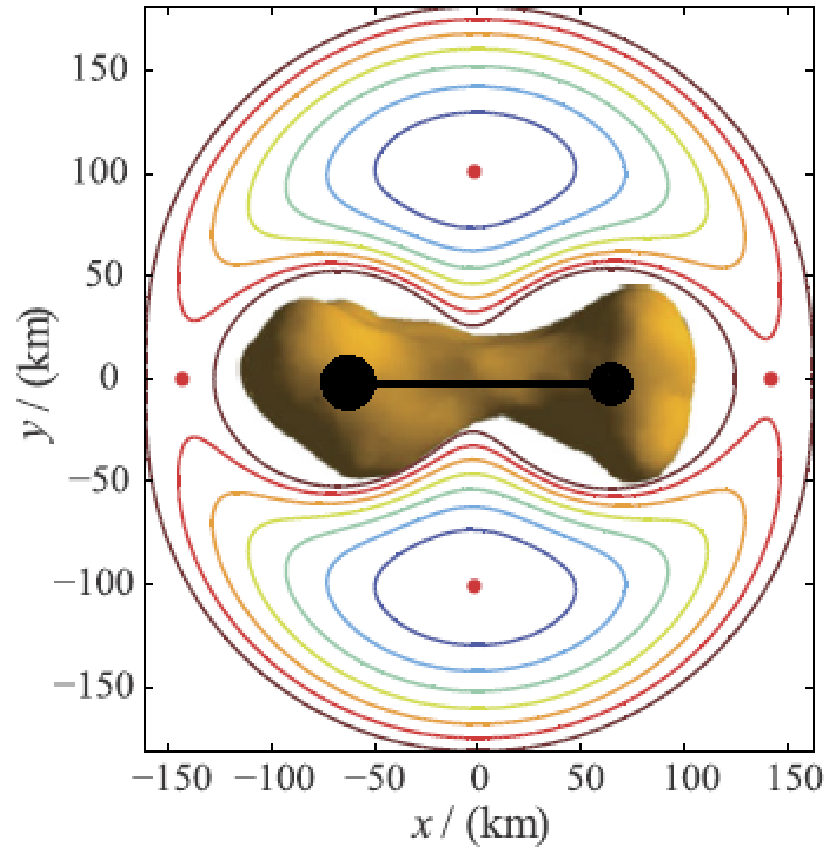
Gravitating Dumbbell



[*] Zeng X., Jiang F., Li J., Baoyin H. Study on the connection between the rotating mass dipole and natural elongated bodies // Astrophysics and Space Science. 2015. V. 356 , no. 1. P. 29–42.

Comparison of approaches





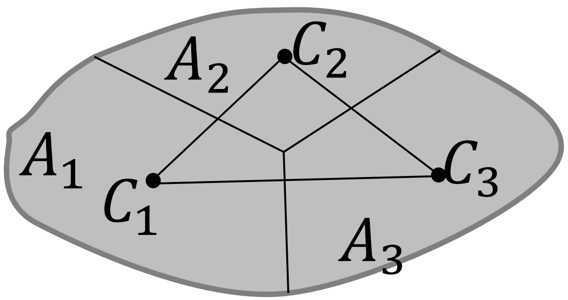
From [*]: m_1 , m_2, l

From [**]: m_1', m_2', l'

$$\delta_1 = \frac{|m_1 - m_1'|}{(m_1 + m_1')/2} \approx \mathbf{0,0654} \qquad \delta_2 = \frac{|m_2 - m_2'|}{(m_2 + m_2')/2} \approx \mathbf{0,0372} \qquad l_2 = \frac{|l - l'|}{(l + l')/2} = \mathbf{0,0432}$$

[*] Burov A.A., Nikonov V.I. et al. On the use of the K-means algorithm for determination of mass distributions in dumbbell-like celestial bodies // Rus. J. Nonlin. Dyn. 2018. V. 14, no. 1. P. 45-52
[**] Zeng X., Jiang F., Li J., Baoyin H. Study on the connection between the rotating mass dipole and natural elongated bodies // Astrophysics and Space Science. 2015. V. 356, no. 1. P. 29–42.

Steinhaus's idea for three subsets



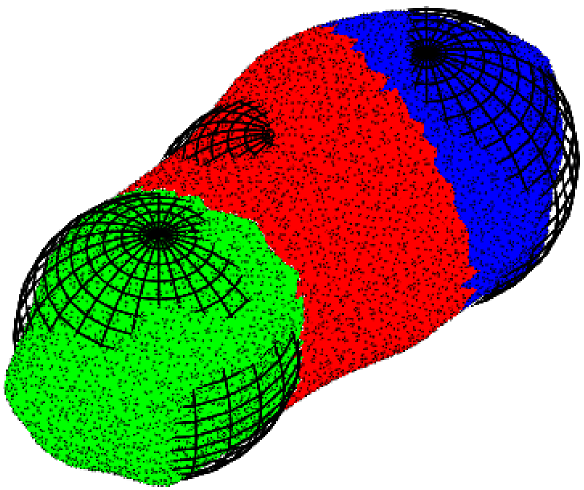
$$
\begin{array}{l|l}
A_1 \to C_1 & \rho_{12} = dist(C_1, C_2) \\
A_2 \to C_2 & \rho_{23} = dist(C_2, C_3) \\
A_3 \to C_3 & \rho_{31} = dist(C_3, C_1)
\end{array}
\Rightarrow \max_{\mathcal{A}} \min_{ij} \rho_{ij},
$$

Choose the one from all possible partitions (A_1, A_2, A_3) such that the minimum of the values ρ_{12}, ρ_{23}, ρ_{31} is the maximum (the minimum of the sides of the triangles is the maximum)

[*] Steinhaus H. Sur la division des corp materiels en parties // Bull. Acad. Polon. Sci. 1956. IV (Cl.III). P. 801–804.

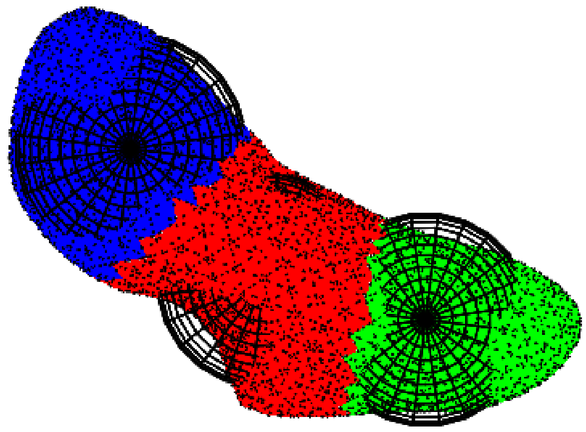
Examples of approximations by three masses

R_k is the radius of the k-th ball,
l_{pq} is the distance between the centers of the p-th and q-th balls.



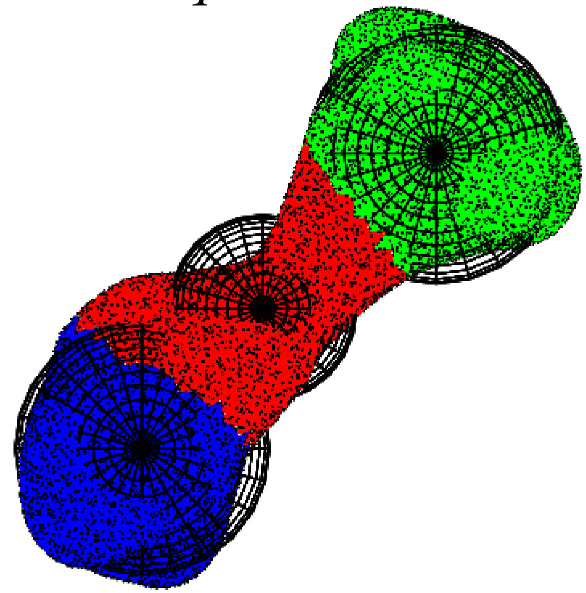
(2063) Bacchus

R_1 =219 m, R_2 =232 m,
R_3 =211 m,
l_{12} =555 m,
l_{23} =264 m,
l_{13} =300 m.



(433) Eros

R_1 =5631 m, R_2 =6151 m,
R_3 =5683 m,
l_{12} =17983 m,
l_{23} =9897 m,
l_{13} =8783 m.

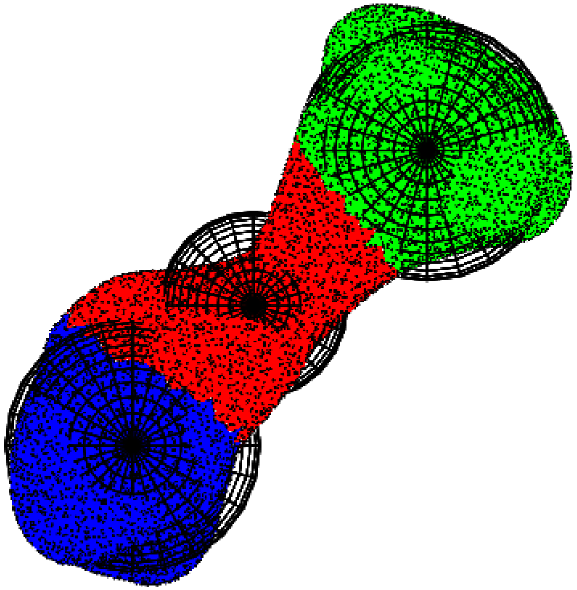


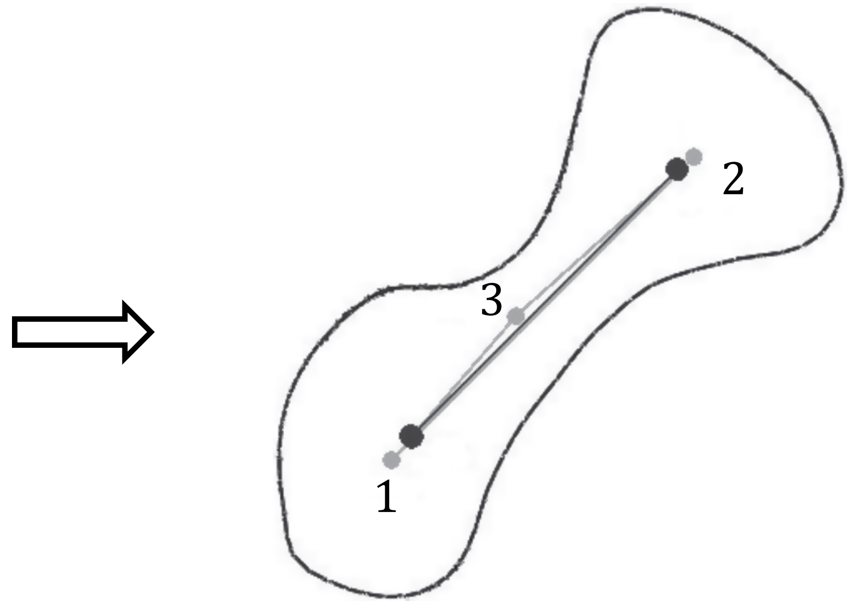
(216) Kleopatra

R_1=41800 m, R_2 =40944 m,
R_3 =30203 m,
l_{12} =133671 m,
l_{23} =59332 m,
l_{13} =74641 m.

[*] Burov A.A., Guerman, A.D., Nikonov V.I. Using the K-Means Method for Aggregating the Masses of Elongated Celestial Bodies // Cosmic Research. 2019. V. 57, no. 4. P. 266–271.

Three-mass approximation for asteroid (216) Kleopatra



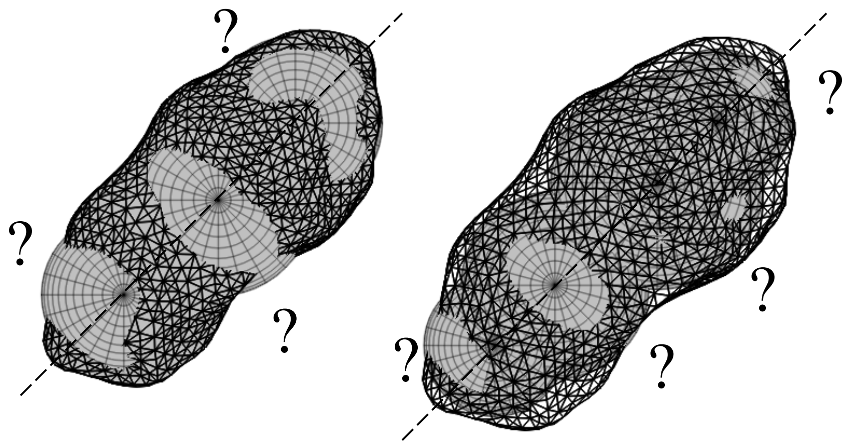


l_{12} =133671 m,

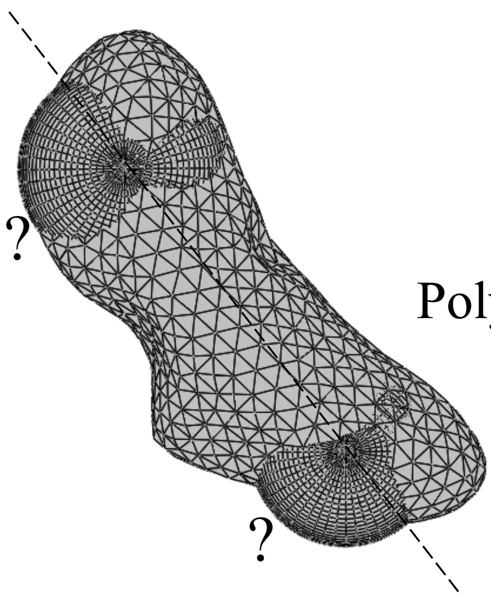
l_{23} =59332 m,

l_{13} =7464 m.

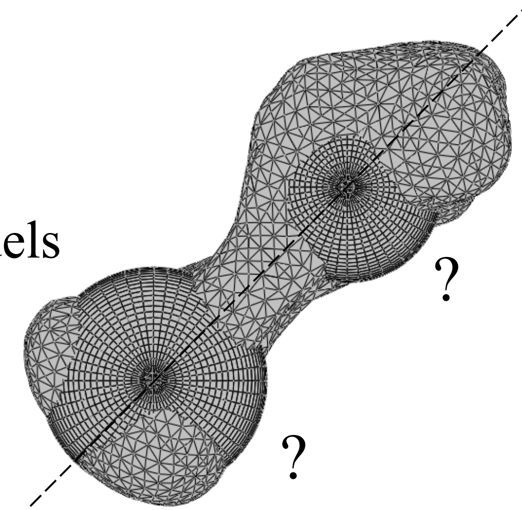
Approximation of gravitational field for almost dynamically symmetric body



(2063) Bacchus.
Polyhedron model and 3- and 4-balls models



(433) Eros.
Polyhedron model and 2-balls model



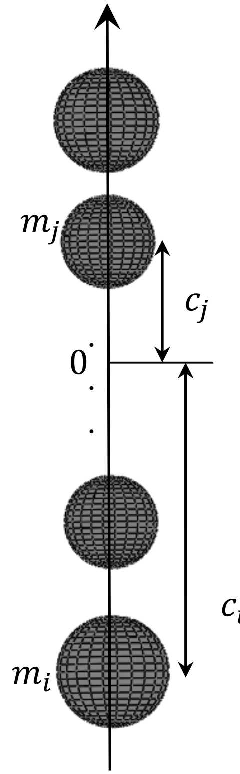
(216) Kleopatra.
Polyhedron model and 2-balls model

Approximation of potential for axisymmetric planet by potential of point mass system

- *Antonov V.A.* Representation of the gravitational field of a planet by the potential of a point mass system // Proc. Astr. Obs. Leningradskii Universitet. 1978, iss. 397, v. 34, p.145-155.
- *Fomin V.N.* On the representation of the gravitational field of the simplest bodies by the attraction of point masses // In: Astronomy and Geodesy, Tomsk: Publishing House of TSU, 1980, iss. 8, p.102-110.
- *Poleshchikov S. M., Kholshevnikov K. V.* Construction of a system of point masses representing the gravitational field of the planet on the basis of satellite observations. I - An algorithm derivation // Leningradskii Univ., Vest., Math, Mech, Astr., 1984, no. 7. P. 76-86.
- *Poleshchikov S. M., Kholshevnikov K. V.* Representation of the geopotential by a system of point masses with complex parameters // Nabliudeniia Iskusstvennykh Sputnikov Zemli, no. 23, 1984, p. 243-248.
- *Poleshchikov S. M., Kholshevnikov K. V.* Refinement of a multipoint potential model on the basis of changes in the orbital elements of satellites // Leningradskii Univ., Vest., Math, Mech, Astr., 1986, p. 87-94.
- *Antonov V. A., Timoshkova E. I., Kholshevnikov K. V.* Introduction to the theory of newtonian potential. Moscow: Nauka. 1988.

The gravitational field of system of balls with centers on one straight line

Gravitational potential of *N* balls with centers on one straight line:



N balls with centers on one straight line

$$U = -G\frac{m'}{r}\left(1 + \sum_{k\geq 1}\frac{\gamma_k}{r^k}P_k(\sin\varphi)\right)$$

$$m' = m_1 + \cdots + m_N,$$

$$\gamma_k = \frac{m_1 c_1^k + \cdots + m_N c_N^k}{m'}$$

φ is latitude of test point

[*] Antonov V. A., Timoshkova E. I., Kholshevnikov K. V. Introduction to the theory of newtonian potential. Moscow: Nauka. 1988.

Calculation of the parameters for the ball system

$$(*)\quad \begin{cases} m_1 + \cdots + m_N = m \ \ (= j_0) \\ m_1 c_1 + \cdots + m_N c_N = mRJ_1 \ \ (= j_1) \\ m_1 c_1^2 + \cdots + m_N c_N^2 = mR^2 J_2 \ \ (= j_2) \\ \cdots \\ m_1 c_1^{2N-1} + \cdots + m_N c_N^{2N-1} = mR^{2N-1} J_{2N-1} \ \ (= j_{2N-1}) \end{cases}$$

System $(*)$ contains $2N$ equations for determining N masses m_i and N coordinates of centers of balls c_i. Here R is radius of sphere circumscribing the body and J_k depends on the bodies mass distribution.

Firstly, a linear subsystem of the first N equations is solved with respect to masses m_i.

Next, substituting found masses into the remaining subsystem of n equations, one obtains a nonlinear system $(**)$ with respect to c_i.

$$(**)\qquad c_i:\ \begin{pmatrix} c_1^N & \dots & c_N^N \\ c_1^{N+1} & \dots & c_N^{N+1} \\ \vdots & \dots & \vdots \\ c_1^{2N-1} & \dots & c_N^{2N-1} \end{pmatrix} \begin{pmatrix} \frac{\det \boldsymbol{V}_1{}'}{\det \boldsymbol{V}'} \\ \vdots \\ \frac{\det \boldsymbol{V}_N{}'}{\det \boldsymbol{V}'} \end{pmatrix} = \begin{pmatrix} j_N \\ j_{N+1} \\ \vdots \\ j_{2N-1} \end{pmatrix}$$

$$\boldsymbol{V}' = \begin{pmatrix} 1 & \dots & 1 \\ c_1 & \dots & c_N \\ \vdots & \dots & \vdots \\ c_1^{N-1} & \dots & c_N^{N-1} \end{pmatrix}, \qquad \boldsymbol{V}_i' = \begin{pmatrix} 1 & \cdots & \overbrace{j_0}^{i} & \cdots & 1 \\ c_1 & \cdots & j_1 & \cdots & c_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1^{N-1} & \cdots & j_{N-1} & \cdots & c_{N-1} \end{pmatrix}$$

Solution of system $(**)$

Let us $\sigma_n = (-1)^{n+1} \sum_{1 \le i_1 < \cdots < i_N \le N} c_1 \cdot \cdots \cdot c_{i_n}$ be elementary symmetric signed polynomials,

then the system $(**)$ can be written in a recurrent form

$$\begin{cases} (\psi_{N,N} =) & j_0\sigma_N + j_1\sigma_{N-1} + j_2\sigma_{N-2} + \cdots + j_{N-1}\sigma_1 = j_N \\ (\psi_{N,N+1} =) & j_1\sigma_N + j_2\sigma_{N-1} + j_3\sigma_{N-2} + \cdots + \psi_{N,N}\,\sigma_1 = j_{N+1} \\ & \cdots \\ (\psi_{N,2N-1} =) & j_{N-1}\sigma_N + \psi_{N,N}\,\sigma_{N-1} + \psi_{N,N+1}\sigma_{N-2} + \cdots + \psi_{N,2N-2}\,\sigma_1 = j_{2N-1} \end{cases}$$

$$\underbrace{\begin{pmatrix} j_0 & j_1 \cdots & j_{N-1} \\ j_1 & j_2 \cdots & j_N \\ j_2 & j_3 \ddots & j_{N+1} \\ \vdots & \vdots \quad \vdots & \vdots \\ j_{N-1} & j_N \cdots & j_{2N-2} \end{pmatrix}}_{\text{Hankel matrix}} \cdot \begin{pmatrix} \sigma_N \\ \sigma_{N-1} \\ \sigma_{N-2} \\ \vdots \\ \sigma_1 \end{pmatrix} = \begin{pmatrix} j_N \\ j_{N+1} \\ j_{N+2} \\ \vdots \\ j_{2N-1} \end{pmatrix}, \quad j_k = mR^k J_k$$

Calculation of parameters J_k for rigid body

Decomposition of the gravitational potential for a rigid body:

$$U_{body} = -G \sum_{d=0}^{\infty} \sum_{a+b+c=d} I_{abc} \quad \frac{(-1)^d}{a!\,b!\,c!} \frac{\partial^d}{\partial x^a \partial y^b \partial z^c} \left(\frac{1}{r}\right)$$

$r = \sqrt{x^2 + y^2 + z^2}$

a, b, c – positive integer,

$I_{abc} = \iiint_{body} X^a Y^b Z^c dm$ – moments of mass distribution of the order *d*.

Calculation of the parameters J_k for rigid body

Decomposition of the gravitational potential for a rigid body:

$$U_{body} = -G\sum_{d=0}^{\infty}\sum_{a+b+c=d} I_{abc}\ \frac{(-1)^d}{a!\,b!\,c!}\frac{\partial^d}{\partial x^a \partial y^b \partial z^c}\left(\frac{1}{r}\right)$$

Let us suppose, the moments of inertia of the body relative to the axis *OX* and *OY* are close. Make a change of variables:

$$x = r\cos\varphi\cos\theta$$
$$y = r\cos\varphi\sin\theta$$
$$z = r\sin\varphi$$

Calculate the average expression:

$$U = \frac{1}{2\pi}\int_0^{2\pi} U(r,\varphi,\theta)\,d\theta = -G\frac{m}{r}\left(1 + \sum_{k=2,\dots,7} J_k\left(\frac{R}{r}\right)^k P_k(\sin\varphi) + \cdots\right)$$

Calculation of the parameters J_k for rigid body

$$J_0 = 1$$

$$J_1 = \frac{I_{001}}{I_{000}\,R}$$

$$J_2 = \frac{1}{I_{000}\,R^2}\left(I_{002} - \frac{1}{2}(I_{020} + I_{200})\right)$$

$$J_3 = \frac{1}{I_{000}\,R^3}\left(I_{003} - \frac{3}{2}(I_{021} + I_{201})\right)$$

$$J_4 = \frac{1}{I_{000}\,R^4}\left(I_{004} - 3(I_{022} + I_{202}) + \frac{3}{8}(I_{040} + I_{400} + 2I_{220})\right)$$

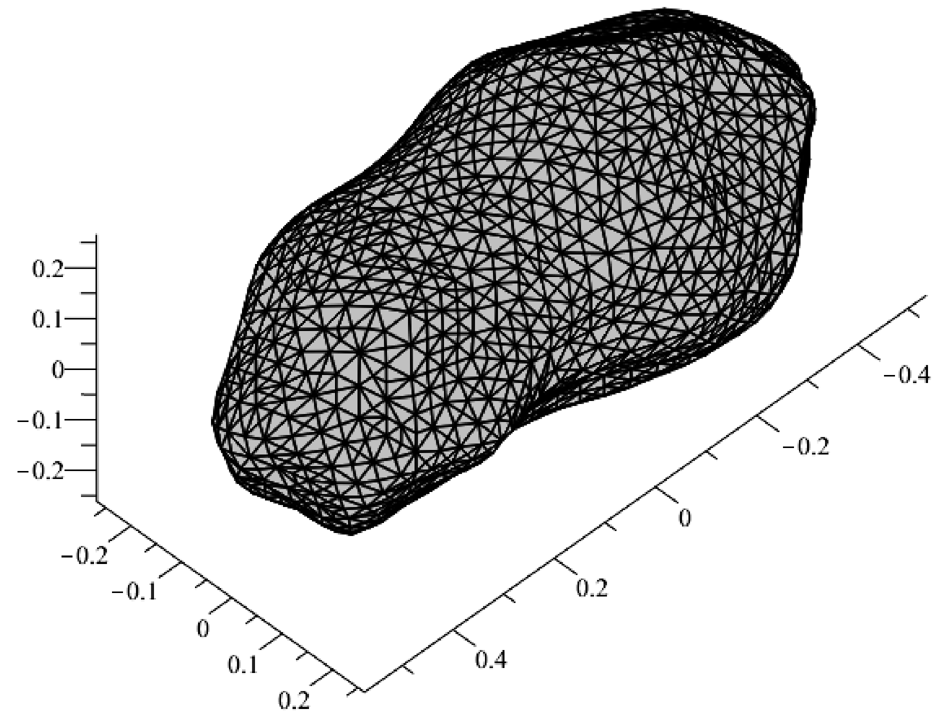
$$J_5 = \frac{1}{I_{000}\,R^5}\left(I_{005} - 5(I_{023} + I_{203}) + \frac{15}{8}(I_{041} + I_{401} + 2I_{221})\right)$$

$$J_6 = \frac{1}{I_{000}\,R^6}\left(I_{006} - \frac{15}{2}(I_{024} + I_{204}) + \frac{45}{8}(I_{042} + I_{402} + 2I_{222}) - \frac{5}{16}\big(I_{060} + I_{600} + 3(I_{240} + I_{420})\big)\right)$$

$$J_7 = \frac{1}{I_{000}\,R^7}\left(I_{007} - \frac{21}{2}(I_{025} + I_{205}) + \frac{105}{8}(I_{043} + I_{403} + 2I_{223}) - \frac{35}{16}\big(I_{061} + I_{601} + 3(I_{241} + I_{421})\big)\right)$$

Example of dynamically symmetric body approximation: asteroid (2063) Bacchus

Asteroid (2063) Bacchus can be considered as almost dynamically symmetrical body:
I_x=6.4356 $\cdot$ 10^9 (kg/km^2)
I_y=20.6365 $\cdot$ 10^9 (kg/km^2)
I_z=20.8344 $\cdot$ 10^9 (kg/km^2)

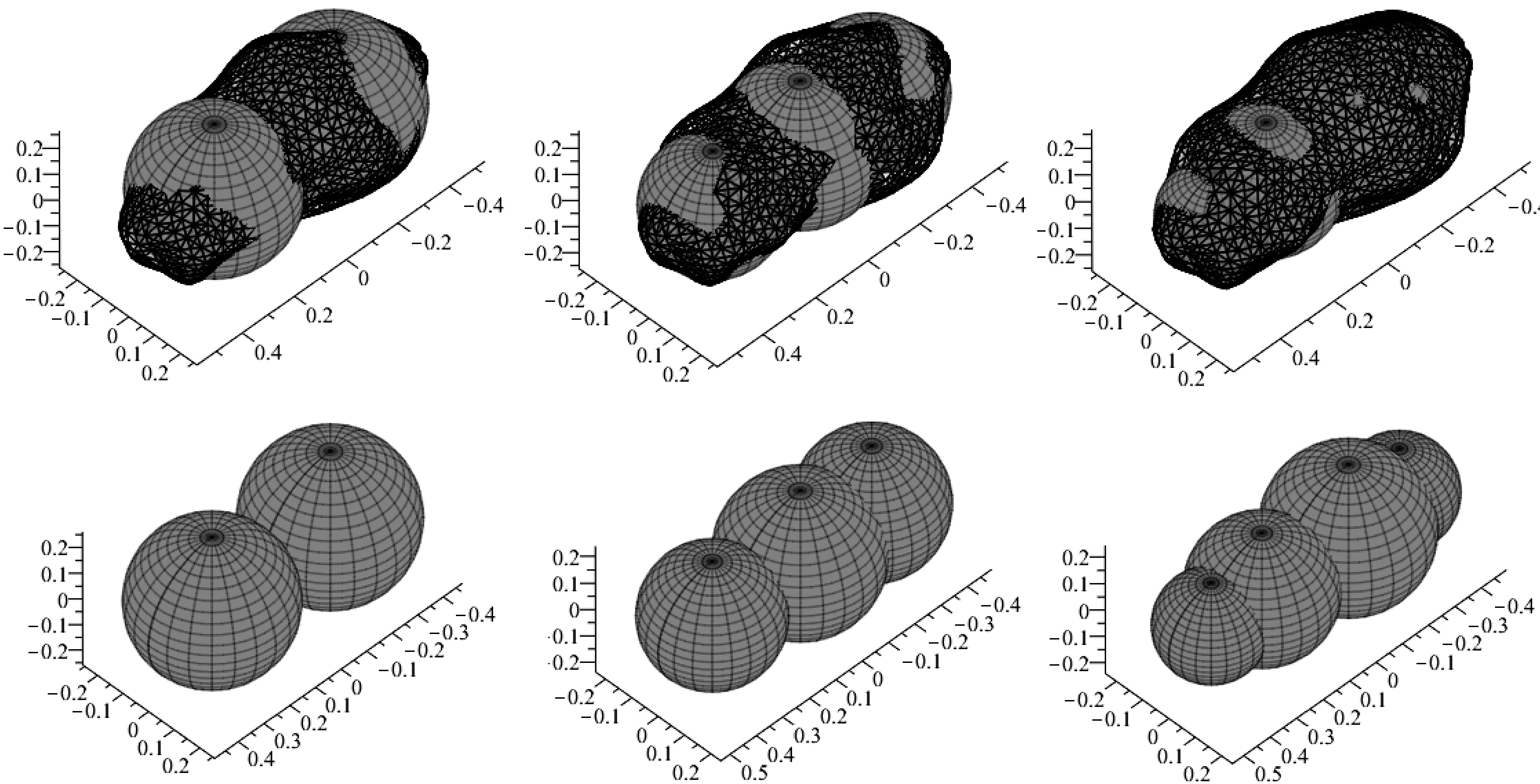


(2063) Bacchus

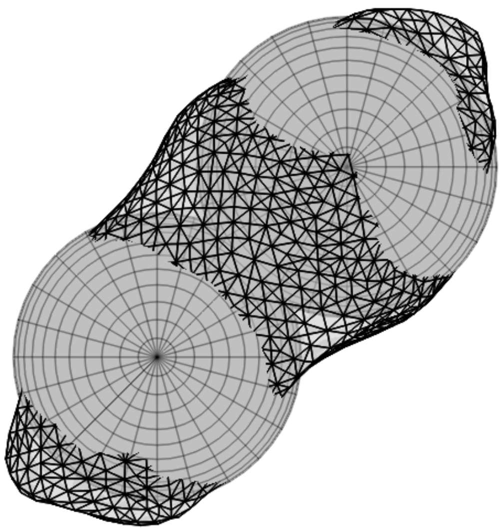
[*] Benner L.A.M., Hudson R.S., Ostro S. J. et. al. Radar Observations of Asteroid 2063 Bacchus // Icarus. 1999. V. 139, no. 2. P. 309-327.

Example of dynamically symmetric body approximation: asteroid (2063) Bacchus

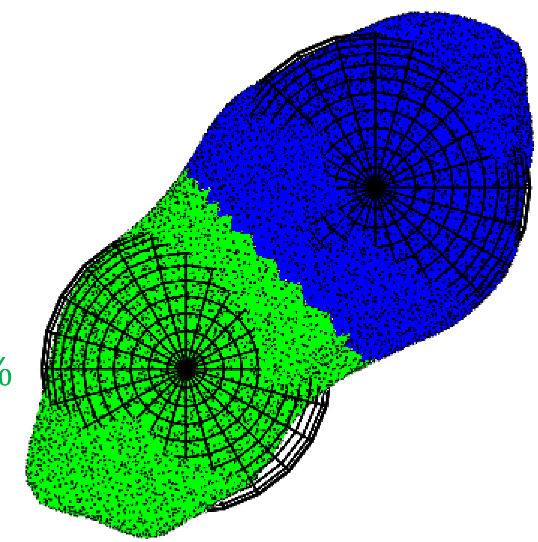
Two, three and four balls approximation for mass distribution of asteroid (2063) Bacchus



Comparison of approaches



Distance between the centers of the balls		
460	438	≈ **5%**
Radius of the first ball (m)		
247,5	244	≈ **1.4%**
Radius of the second ball (m)		
258	261	≈ **1.15%**



K-means method [*]

[*] Burov A.A., Nikonov V.I. et al. On the use of the K-means algorithm for determination of mass distributions in dumbbell-like celestial bodies // Rus. J. Nonlin. Dyn., 2018, Vol. 14, no. 1, pp. 45-52

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Thanks for your attention!