Finite-point approximations of fields of attraction and their verification

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Abstract. Approaches to constructing finite-point approximations of the gravitational fields of celestial bodies with complex shapes that are far from spherical are discussed. The study shows good agreement between the parameters of body mass distribution models obtained using both the K-means method and a system of balls with centers on a straight line.

Introduction

In modern celestial mechanics, a lot of attention is paid to the study of motion in the vicinity of small celestial bodies, in particular, asteroids and comets. The actuality of the related issues is provided by the intensification of research in connection with the problem of asteroid danger, as well as with the design and implementation of missions to work both in the vicinity of such celestial bodies and on their surface.

As is known (see, e.g., [1, 2, 3, 4, 5, 6]), celestial mechanics relies on the development of the potential energy of Newton's gravitational attraction into a series. Such a development is based on a natural small parameter, expressing the ratio of the characteristic sizes of the attracting bodies to the distance between them. The second-order truncation is usually sufficient to accurately describe and predict the dominant dynamic effects of mutual attraction. However, numerous small celestial bodies are of a complex shape. At the same time, many small celestial bodies have a rather complex shape. The correct description of the fields of attraction generated by them requires the use of higher approximations. Currently, the so-called Werner-Scheeres approach is of the most widely used. Its main provisions of which are set out in publications [7, 8]. The Werner-Scheeres approach is effective for numerical calculations in the case when a celestial body is assumed to be homogeneous or, more generally, such a body can be represented as a set of homogeneous disjoint components. Assuming that the surface of a small celestial body is defined by a triangulation grid, the Werner-Scheeres method allows us to represent its potential of attraction as the sum of the potentials of individual tetrahedra, with a common vertex and bases in the triangulation cells. This sum consists of a large number of terms. In an order this number is comparable to the number of elements of the graph defining the triangulation grid. It is clear that such an approximation is essentially unsuitable for a preliminary analytical study of motion in the vicinity of a small celestial body. In this regard, the problem of constructing a system of equigravitating bodies seems being very important. For such a system, the components of the Euler-Poinsot tensor, otherwise known as inertia integrals, must coincide with the corresponding components of such a tensor for the initial body for the highest possible order.

The problem of the approximation of the field of attraction of a celestial body by the field of attraction of a set of "elementary" bodies is the subject of this study. The goal is to find such approximate configurations for which the components of the Euler-Poinsot tensor will coincide with the corresponding components for the initial body not only for the second, but also for a higher order. The results obtained are compared with the results obtained earlier using the K-means method, applied in conditions where the assumptions of the Werner — Scheeres theorem on the approximation of the potential of a body are valid. As examples, models of a number of small celestial bodies are considered.

1. On K-means method

H. Steinhaus [9] proposed a novel approach to dividing sets of points into non-overlapping groups, which is widely used in the field of pattern recognition. Let A be a set of a finite number of points located in three-dimensional Euclidean space in some way. Let $A_1, \ldots, A_k$ be disjoint subsets of A such that their union is exactly equal to A:

$$A = A_1 \cup \cdots \cup A_k, \quad A_i \cap A_j = 0, \; i \neq j. \tag{1}$$

Let $S_1, \ldots, S_k$ be the centroids of these subsets and $\rho_{ij} = |S_i S_j|$ be the pairwise distance between them. Let us denote $\rho = \min\limits_{i \neq j} \rho_{ij}$ as the minimum distance between centroids.

According to Steinhaus [9], we are looking for a partition of the set A that satisfies the requirements (1) such that the minimum distance between the centroids of the subsets is maximised: $\rho_\star = \max\limits_{A} \rho$. If such a partition exists, it is said that the subsets in the partition are "as far apart as possible". Since the iteration is finite, there is at least one partition of A into disjoint subsets that achieves the maximum $\rho_\star$, and this partition is not necessarily unique.

Lloyd [10] proposed an iterative algorithm that approximates $\rho_\star$. The algorithm never repeats splitting, and it is guaranteed to converge, see, e.g., [10, 11].

Suppose that the surface of a body can be represented as a polyhedron, which consists of a given set of vertices and a set of triangular faces that are consistently oriented. The application of the Steinhaus approach and Lloyd's algorithm to the centroids of these tetrahedra equipped with corresponding oriented volumes ([12]) made it possible to construct two-, three- and four-point approximations for asteroids (2063) Bacchus, (216) Kleopatra, (433) Eros, (1620) Geographos and comet (67P) Churyumov-Gerasimenko [13, 14, 15]

2. Comparison with finite-point approximations obtained otherwise

According to [14], the K-means method defines a triple of points P_1', P_2' and P_3' with masses $m_1' = 2.001 \cdot 10^{15}$, $m_2' = 2.608 \cdot 10^{15}$ and $m_3' = 2.057 \cdot 10^{15}$ kg, respectively. At the same time $|P_1'P_3'| = 17.983$, $|P_2'P_3'| = 9.897$, $|P_1'P_2'| = 8.783$. The triangle $\Delta P_1'P_2'P_3'$ is obtuse, with an obtuse angle $\angle P_1'P_2'P_3' = 2.592524415$ rad, close to the straight one.

On the other hand, the Grebenikov-Demin-Aksenov method (see, e.g., [16]) gives a triple of collinear points P_1, P_2 and P_3 with masses $m_1 = 1.656 \cdot 10^{15}$ kg, $m_2 = 2.696 \cdot 10^{15}$ kg and $m_3 = 2.313 \cdot 10^{15}$ kg. At the same time $|P_1P_3| = 19.896$ km, $|P_2P_3| = 10.312$ km, $|P_1'P_2'| = 9.584$ km.

Mass discrepancies amounting to

$$\frac{|m_1 - m_1'|}{\min(m_1, m_1')} \approx 0.2083, \quad \frac{|m_2 - m_2'|}{\min(m_2, m_2')} \approx 0.0337, \quad \frac{|m_3 - m_3'|}{\min(m_3, m_3')} \approx 0.1245,$$

does not exceed 21 percent.

Similarly calculated differences in distances amounting to

$$\delta_{12} \approx 0.106, \ \delta_{23} \approx 0.042, \ \delta_{13} \approx 0.091, \ \delta_{ij} = \frac{\big||P_i'P_j'| - |P_iP_j|\big|}{\min(|P_i'P_j'|, |P_iP_j|)},$$

does not exceed 11 percent.

It remains to be noted that Steinhaus' approach is purely geometric. Its use does not imply at least some knowledge about the gravitational potential of the studied celestial body.

The study shows a good agreement between the parameters of body mass distribution models obtained using both the K-means method and a system of balls placed along a straight line.

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