Network of families of symmetric spatial periodic orbits in the Hill problem via symplectic invariants

Cengiz Aydin and Alexander Batkhin

Abstract. A technique of Conley–Zehnder indices is applied for investigation of interconnections of the basic families of periodic orbits with maximal numbers of symmetries of the well-known Hill problem. These basic families are g, f – families of planar direct and retrograde periodic orbits, and $\mathcal{B}_0$ – family of rectilinear vertical consecutive collision orbits. The relations among families of periodic orbits are provided by families of spatial symmetric periodic orbits which makes k-covering at the bifurcation points. All the families form a common network and can be represented as well-organized bifurcation graphs of the interconnectedness.

Introduction

1. Circular Hill Problem, its symmetries and basic families

The Hill three-body problem (Hill3BP), a limiting case of the circular restricted three-body problem (RTBP), is a well-known model which provides an approximation of the dynamics of the infinitesimal body in the vicinity of the smaller primary. In its original application, George Hill reformulated the lunar theory and discovered a periodic solution with period equal to the synodic month of the Moon. There are a lot of applications of Hill's approach such as capturing in the dynamics of natural or artificial satellites, distant moons of asteroids, low-energy escaping trajectories, frozen orbits around planetary satellites. Hill3BP problem's periodic solution can be continued to RTBP or even into three-body problem solutions and thus can be used in astrodynamical projects.

Hill3BP problem Hamiltonian

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) - \frac{1}{r} + p_x y - p_y x - x^2 + \frac{1}{2}\left(y^2 + z^2\right), \quad (1)$$

where $r = \sqrt{x^2 + y^2 + z^2}$, consists of the rotating Kepler problem Hamiltonian with a velocity independent gravitational perturbation produced by the massive primary (the quadratic form of x, y, z). This difference between the rotating Kepler problem and Hill3BP system gives a dramatic dynamical change. While the rotating Kepler problem is an integrable system, the Hill 3BP is non-integrable. Equations of motion derived from (1) are invariant under discrete group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of symplectic (anti-symplectic) symmetries ρ of the extended phase space:

$$\rho(\alpha, \beta, \gamma) : (t, x, y, z, p_x, p_y, p_z) \to (\alpha t, \beta x, \alpha\beta y, \gamma z, \alpha\beta p_x, \beta p_y, \alpha\gamma p_z), \qquad (2)$$

where $\alpha, \beta, \gamma \in \{+1, -1\}$. So all solutions to the Hill3BP can be divided into groups with different number of symmetries (2). The symmetry of a periodic solution plays an essential role, since it allows one to investigate it numerically for only parts of the period.

There are 3 families, called *basic families*, whose orbits are simple and have the largest number of symmetries: g and f are families of planar direct and retrograde satellite orbits [4] and $\mathcal{B}_0$ is a family of vertical collision orbits [5]. The last one consists of two branches called $\mathcal{B}_0^+$ and $\mathcal{B}_0^-$ for upper and lower coordinate subspaces correspondingly. Other important families are Lypunov families a and c emanating from the librations points L_1 and L_2 and family g' appeared after symmetry breaking bifurcation of the family g [4]. It was shown in [3] that all these families are connected to each other by families of spatial periodic orbits and form a kind of common network. Current work significantly extends these results by systematically applying the technique of Conley–Zehnder indices.

2. On Conley–Zehnder indices μ_{CZ} of periodic solution

The Conley–Zehnder index μ_{CZ} assigns a mean winding number to non-degenerate periodic orbits, which stays constant until a bifurcation point is achieved. In its formal definition, the index μ_{CZ} is associated with a path of symplectic matrices generated by the linearized flow along the whole orbit. This path starts at the identity and ends at the reduced monodromy matrix whose Floquet multipliers are different from 1 due to the non-degeneracy of the orbit. The index μ_{CZ} measures the twisting of this symplectic path by counting the number of crossing the eigenvalue 1. If the orbit becomes degenerate, i.e., 1 is among its Floquet multipliers, then bifurcation appears and the index jumps according to direction of crossing the eigenvalue 1. For instance, if a pair of elliptic Floquet multipliers in the form $e^{\pm i\theta}$ becomes positive hyperbolic, then the corresponding index jump depends on whether the eigenvalue 1 is crossed from above (i.e., by $e^{i\theta}$) or from below (i.e., by $e^{-i\theta}$). In one case the index jumps down and in the other case the index jumps up. In order to determine this direction of crossing the eigenvalue 1 we consider the Krein signature (especially its version for symmetric periodic orbits) which specifies the direction of the rotation and thereby the index jump.

When working locally near a family of non-degenerate periodic orbits, then there is a fascinating bifurcation-invariant: the local Floer homology and thus its Euler characteristic, the alternating sum of the ranks of the local Floer homology groups. Significantly, the index leads to a grading on local Floer homology and thus, the index provides important information how different families are related to each other before and after bifurcation.

We use these symplectic invariants to construct bifurcation graphs in the same way as introduced in [2], where networks of families of symmetric spatial periodic orbits associated to g, g' and f, and their multiple cover bifurcations, were demonstrated. A "bifurcation graph" is a labelled graph, whose vertices correspond to bifurcation points and whose edges correspond to families of periodic orbits, labelled with their Conley–Zehnder index (see Figure 1 for an example). This approach provides additional structure to the families of periodic orbits and supports to examine their connections at bifurcation points from a topological point of view. In particular, this allows to check at every bifurcation point the Euler characteristics before and after bifurcation 1. In case the Euler characteristics do not coincide, then there are still undiscovered families at this bifurcation point.

Instead of using the formal definition to determine the indices, we follow the approach developed in [1, 2], in which the indices are known via analytical considerations in view of the origin of the families. For very low energies, the regularized Kepler problem is the source of the families g, f and $\mathcal{B}_0^{\pm}$. Notice that planar orbits have planar and spatial indices, μ_{CZ}^p and μ_{CZ}^s. It was shown [1] that their indices are given by

$$\mu_{CZ} = \begin{cases} 6 = \mu_{CZ}^p + \mu_{CZ}^s = 3+3 & \text{for family } g \\ 4 & \text{for family } \mathcal{B}_0^{\pm} \\ 2 = \mu_{CZ}^p + \mu_{CZ}^s = 1+1 & \text{for family } f. \end{cases}$$

We start with these indices, continue those families for higher energies, follow their Floquet multipliers together with corresponding Krein signatures, examine the interaction of μ_{CZ} with bifurcation points and construct bifurcation graphs, such as shown in Figure 1.

3. Interconnections between the basic families

The purpose in our study is to provide bifurcation graphs showing rich connections between basic families of periodic orbits and their bifurcations. To be emphasized is that our investigations show the following structures of bifurcation results of families of spatial orbits in each row (in each row the integer n indicates each n-th cover bifurcation of the underlying family in the first row):

g	g'	$\mathcal{B}_0^{\pm}$	f	f_3	halo
	1				2
1		2			
2	2	3			
3	3	4	5	1	
4	4	5	6	2	
5		6	7		

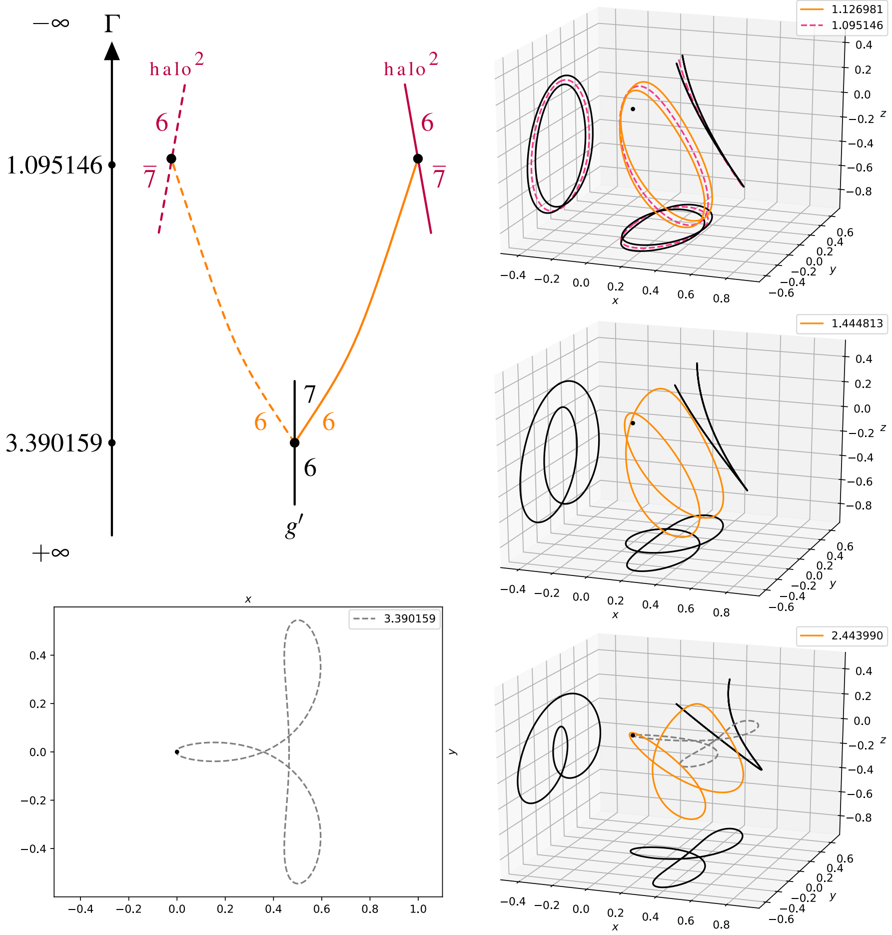

FIGURE 1. Left top: Bifurcation graph associated to connection between g' and double cover of halo orbit (denoted by halo^2). Corresponding orbits start bottom left, then right, then up.

As a consequence, we have discovered connections at bifurcation points between n-th cover of the families g, $n+1$-th cover of $\mathcal{B}_0^{\pm}$ and $n+2$-th cover of f, for $n = 3, 4, 5$. Such pattern can be expected in view of their Conley–Zehnder indices, which play a significant role in this paper. In particular, this work aims to demonstrate that the technique of such symplectic invariants supports to deduce such connections at bifurcation points which are hard to see by bare computations.

One example of a bifurcation graph is shown in Figure 1, which shows the connection in the first row from the previous overview, i.e., between g' and double cover of halo orbits. Let us verify that the corresponding bifurcation points in Figure 1 are in accordance with the Euler characteristics before and after bifurcation. At $\Gamma = 3.390159$ the Euler characteristics before and after bifurcation are

$$(-1)^6 = 1, \quad 2 \cdot (-1)^6 + (-1)^7 = 1.$$

At $\Gamma = 1.095146$ the Euler characteristics are $(-1)^6 = 1$ before and after bifurcation. Notice that the index $\overline{7}$ indicates bad orbits, which are ignored in the local Floer homology and not counted.

References

[1] Aydin C. From Babylonian lunar observations to Floquet multipliers and Conley–Zehnder Indices. *J. Math. Phys.* **64**(8) (2023) https://doi.org/10.1063/5.0156959

[2] Aydin C. The Conley–Zehnder indices of the spatial Hill three-body problem. *Celest. Mech. Dyn. Astron.* **135**(32) (2023) https://doi.org/10.1007/s10569-023-10134-7

[3] Batkhin A.B. and Batkhina N.V. Hierarchy of periodic solutions families of spatial Hill's problem. *Solar System Research.* **43**(2) (2009), pp. 178–183. https://doi.org/10.1134/s0038094609020105.

[4] Hénon M. Numerical exploration of the restricted problem V. Hill's case: periodic orbits and their stability. *Astron. & Astrophys.* **1** (1969), pp. 223–238.

[5] Lidov M.L. A method of construction of families of spatial periodic orbits in Hill's problem. *Kosmicheskie Issledovaniia.* **20**(6) (1982), pp. 787–807, (in Russian).

Cengiz Aydin
Institut für Mathematik, Universität Heidelberg, Germany
e-mail: cengiz.aydin@hotmail.de

Alexander Batkhin
Department of Aerospace Engineering, Techion – Israel Institute of Technology, Haifa, Israel
e-mail: batkhin@technion.ac.il