On the Evolution of Asteroid Orbit in the Restricted Circular Three-Body Problem: External and Internal Cases, New Results

Pavel Krasilnikov and Alexander Dobroslavskiy

Abstract. The spatial circular restricted three-body problem in the nonresonant case is investigated. We apply Gaussian averaging to obtain averaged equations of motion in terms of osculating elements. A Keplerian ellipse with a focus at the main body (the Sun) is taken as an unperturbed orbit. We derive a twice-averaged disturbing function in the form of an explicit analytical series with coefficients that are expressed in terms of Gauss and Clausen hypergeometric functions. For a reduced system, phase portraits of oscillations in the plane of are shown in the fourth approximation. The radius of convergence of the power series for fixed values of Lidov-Kozai integral was investigated. It is shown that the power series is asymptotic in the sense of Poincaré in the regions of divergence. The asymptotic nature of the series allows the use of perturbation theory methods in regions of divergence, excluding uniformly close orbits. An estimate of the number of retained members of the series is obtained, which guarantees the reliability of constructing phase portraits.

Introduction

We investigate the classical problem of the Keplerian orbit evolution for a massless body in the gravitational field of two primaries (the Sun and Jupiter). This problem was first considered by Gauss in 1809. Zeipel [1] continued these studies by investigating Lindstedt series of solutions to the problem. A detailed study of Hill's case is contained in the articles [2, 3]. The main goal of the report is to obtain new results using modern information technologies.

1. Statement of the problem

We consider the circular spatial restricted three-body problem. Assume that a massless body (asteroid, or satellite) P is in the gravitational field of two primaries moving in a circular orbit of radius r_J. The central body S (Sun) of mass m_S affects the asteroid with the force F_J, and the second body J (Jupiter) of mass m_J has a disturbing effect with the force F_J. Assume that the unperturbed trajectory of the satellite is a Keplerian ellipse with a focus at S, and its plane Π makes an angle of i with the plane Π_0 of motion of the attracting bodies (Fig. 1).

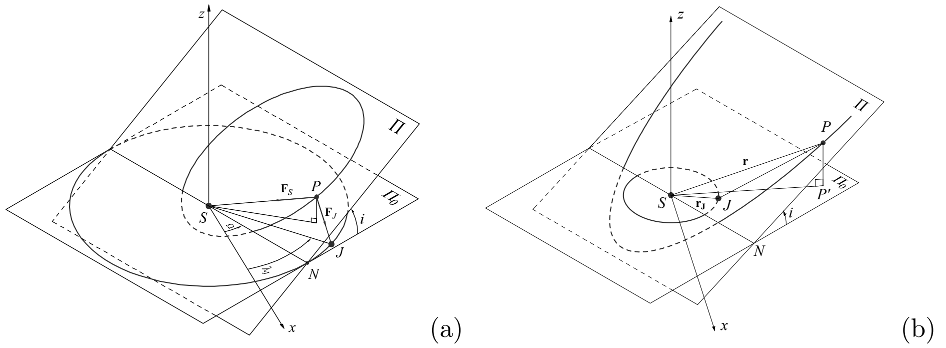


FIGURE 1. Internal (a) and external (b) cases

2. Averaged perturbation function

The perturbation functions of the problem and their twice averaging are the following:

Internal case	External case
$R=\dfrac{fm_J}{r_J}\sum\limits_{n=2}^{\infty}\left(\dfrac{r}{r_J}\right)^n P_n(\cos\gamma),$	$R=\dfrac{fm_J}{r}\left(1+\dfrac{r_J^3-r^3}{r_J^2 r}+\sum\limits_{n=2}^{\infty}\left(\dfrac{r_J}{r}\right)^n P_n(\cos\gamma)\right),$
$R^{**}=\dfrac{fm_J}{r_J\sqrt{1-e^2}}\sum\limits_{n=1}^{\infty}D_n\left(\dfrac{a}{r_J}\right)^{2n},$	$R^{**}=\dfrac{fm_J}{a}\left(1+\dfrac{1}{\sqrt{1-e^2}}\sum\limits_{n=1}^{\infty}D_n\left(\dfrac{r_J}{a}\right)^{2n}\right),$
$D_n=(1+e)^{2n+2}P_{2n}(0)\times \left(F_{2,1}\left(\dfrac{1}{2},2n+2;1;\dfrac{2e}{e-1}\right)\times P_{2n}(0)P_{2n}(\cos i)+ 2\sum\limits_{k=1}^{n}(-1)^k A_{2k}^{(2n)}(e,i)\cos 2k\omega\right).$	$D_n=(1+e)^{1-2n}P_{2n}(0)\times \left(F_{2,1}\left(\dfrac{1}{2},1-2n,1;\dfrac{2e}{e-1}\right)\times P_{2n}(0)P_{2n}(\cos i)+ 2\sum\limits_{k=1}^{n}A_{2k}^{(2n)}(e,i)(-1)^k\cos 2k\omega\right).$

Here $r=\dfrac{a\left(1-e^2\right)}{1+e\cos\nu}$, γ is the angle between r_J and r, $P_n(\cos\gamma)$ is the Legendre polynomial, $F_{2,1}$ and $F_{3,2}^{reg}$ is the Gaussian and Clausen functions.

3. Phase portraits of oscillations in a reduced system

We have three first integrals of the evolution equations:

$$a = c_0, \qquad \left(1 - e^2\right)\cos^2 i = c_1, \qquad R^{**} = h$$

The reduced equations hare the following

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \hat{R}}{\partial \omega}, \qquad \frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \hat{R}}{\partial e}, \qquad \hat{R} = R^{**}|_{(1-e^2)\cos^2 i = c_1}$$

Phase portraits of oscillations in the fourth approximation ($n = 4$) are shown in Fig.2

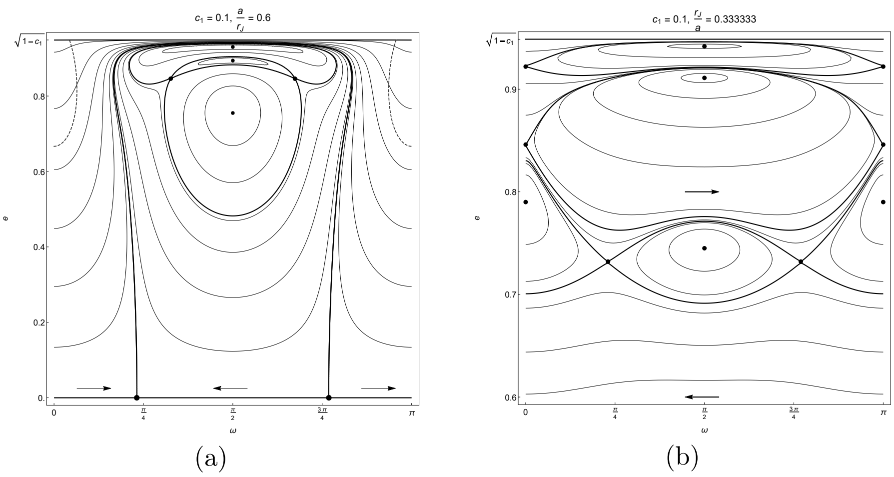


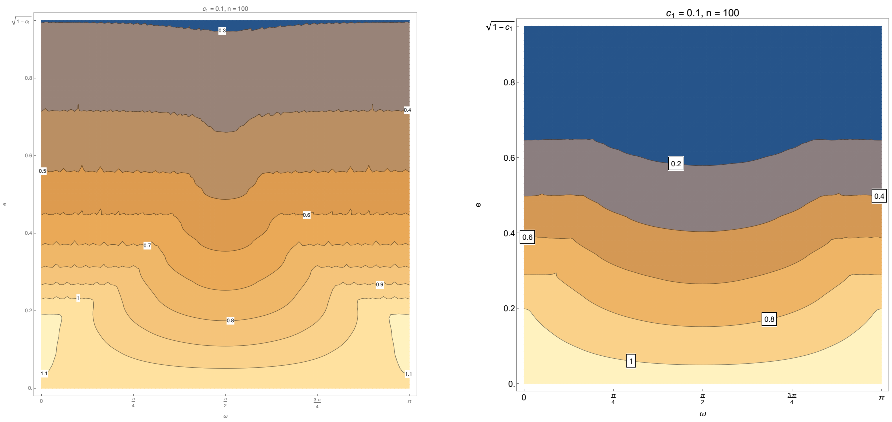
FIGURE 2. Phase portraits (a) internal case $c_1 = 0.1$, $a/r_J = 0.6$, (b) external case $c_1 = 0.1$, $a/r_J = 0.333$

4. Convergence and divergence regions of power series of averaged perturbation function

The convergence radius of function $\hat{R}(a, e, \omega, c_1)$ is calculated using the Cauchy-Hadamard formula:

$$\rho\left(e, \omega, c_1\right) = \left(\overline{\lim_{n\to\infty}} \sqrt[n]{|D_n|}\right)^{-1}$$

The curves isolines $\rho\left(e, \omega, c_1\right) = const$ in plane (e, ω) for $c_1 = 0.1$ and $n = 100$ are shown in the following figures [4]. The power series of $\hat{R}(a, e, \omega, c_1)$ diverges above the curve $\rho\left(e, \omega, c_1\right) = \mu$ when μ is the parameter of expansion. Below this curve, the series converges.



(a) (b)

FIGURE 3. Convergence radius (a) internal case, (b) external case

5. On Poincaré asymptoticity of a power series

We investigated [4] the behavior of a power series in regions of divergence. It is shown numerically that this series is asymptotic in the sense of Poincaré, i.e.

$$\left\| \hat{R} - \hat{R}_k \right\| \sim O\left(\varepsilon^{k+1}\right)$$

over a finite period of time where $\hat{R}_k$ is partial sum of a series. Here k is the number of retained members of the series. It follows from the calculations that the partial sum of seventy terms approximates the function with high accuracy. The asymptotic nature of the series allows, using traditional methods of perturbation theory, to study the evolution of Keplerian orbital elements for all values of μ from the interval $[0, 1)$, excluding the case $\mu \approx 1$.

References

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Pavel Krasilnikov
Department of Theoretical Mechanics and Mechatronics,
Moscow Aviation Institute (National Research University),
Moscow, Russia
e-mail: krasil06@rambler.ru

Alexander Dobroslavskiy
Department of Theoretical Mechanics and Mechatronics,
Moscow Aviation Institute (National Research University),
Moscow, Russia
e-mail: a.dobroslavskiy@gmail.com