Lerman separatrix map in the problem of satellite attitude motion

V.V.Sidorenko

Keldysh Institute of Applied Mathematics RAS,
Moscow Institute of Physics and Technology

Introduction

Three names that determined the development of studies on the rotational motion of celestial bodies

Leonard Euler

Vladimir Beletsky

Jack Wisdom

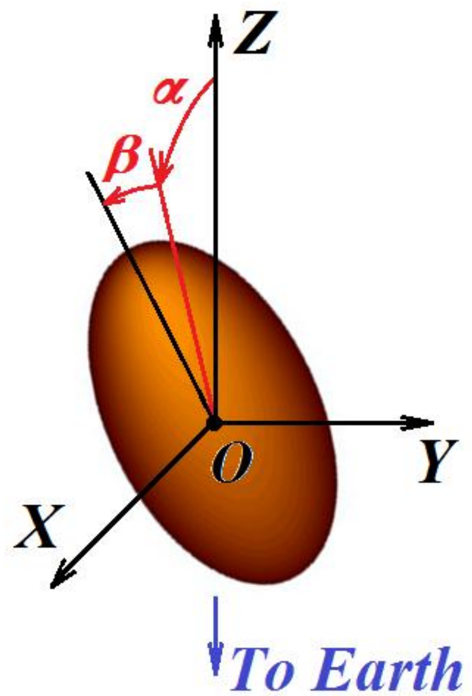
Introduction

The analysis of some problems in attitude dynamics requires the study of 2DOF Hamiltonian systems with a pair of orbits bi-asymptotic to a saddle-center equilibrium. It provides the possibility of applying the approach developed by L.M.Lerman and C. Grotta Ragazzo.

•Lerman L.M. Hamiltonian systems with loops of a separatrix of a saddle-center // Sel. Math. Sov. 1991. V.. 10. P. 297-306.

•Grotta Ragazzo C. On the stability of double homoclinic loops // Commun. Math. Phys. 1997. V. 184. P. 251-272.

Axisymmetric satellite in a gravity field: motion equations



$$\frac{d\alpha}{d\tau}=\frac{\partial H}{\partial p_\alpha},\quad \frac{d\beta}{d\tau}=\frac{\partial H}{\partial p_\beta}$$

$$\frac{dp_\alpha}{d\tau}=-\frac{\partial H}{\partial \alpha},\quad \frac{dp_\beta}{d\tau}=-\frac{\partial H}{\partial \beta}$$

$$H(p_\alpha,p_\beta,\alpha,\beta)=H_0(p_\alpha,\alpha)+H_1(p_\alpha,p_\beta,\alpha,\beta)$$

$$H_0(p_\alpha,\alpha)=\frac{1}{2}[p_\alpha^2+3(\Theta_c-1)\cos^2\alpha],$$

$$H_1(p_\alpha,p_\beta,\alpha,\beta)=\frac{1}{2}\left[p_\beta^2+(p_\alpha+1)^2 tg^2\beta-3(\Theta_c-1)\cos^2\alpha\cos^2\beta\right]$$

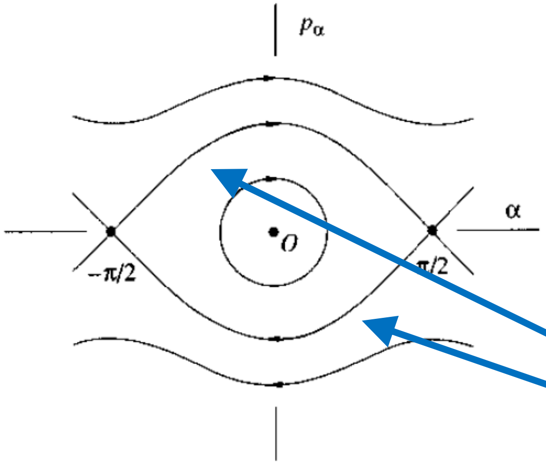
$\Theta_c>1$ – *prolate satellite* $\Theta_c<1$ – *elongated satellite*

If $\mathbf{z}(\tau)=(\alpha(\tau),\beta(\tau),p_\alpha(\tau),p_\beta(\tau))^T$ is a solution of the motion equations,

then $Q\mathbf{z}(-\tau)$, $Q=diag(-1,1,1,-1)$ is a solution too.

Invariant manifold

Phase flow on the invariant manifold $\beta = p_\beta \equiv 0$



$$\Theta_c < 1 - \textit{elongated satellite}$$

$$\alpha_* = \frac{\pi}{2} \bmod \pi, \quad \beta = 0 \bmod \pi$$

Physically different!

The strategy: to use the composition of the approximate local and global maps for studies of the phase flow properties in the vicinity of this double homoclinic loop

Construction of the local map (L-GR)

$\Theta_c < 1 - elongated\ \ satellite$ $\qquad \alpha_* = \frac{\pi}{2} \bmod \pi, \quad \beta = 0 \bmod \pi$

Consequence of Generalized Lyapunoff Theorem (Moser-Russmann)

$$\mathbf{z} = (\alpha, \beta, p_\alpha, p_\beta)^T \mapsto \tilde{\mathbf{z}} = (\tilde{\alpha}, \tilde{\beta}, \tilde{p}_\alpha, \tilde{p}_\beta)^T$$

$$\tilde{p}_\alpha = p_\alpha + O(|\mathbf{z} - \mathbf{z}_*|^2), \quad \tilde{p}_\beta = p_\beta + O(|\mathbf{z} - \mathbf{z}_*|^2)$$

$$\tilde{\alpha} = \alpha - \alpha_* + O(|\mathbf{z} - \mathbf{z}_*|^2), \quad \tilde{\beta} = \beta + O(|\mathbf{z} - \mathbf{z}_*|^2)$$

$$\mathbf{z}_* = (0, 0, \alpha_*, 0)$$

New Hamiltonian

$$\tilde{H} = \kappa I_\alpha + I_\beta + O(I_\alpha^2 + I_\beta^2)$$

$$I_\alpha = \frac{1}{2\kappa}[\tilde{p}_\alpha - (\kappa\tilde{\alpha})^2] \qquad I_\beta = \frac{1}{2}(\tilde{p}_\beta^2 + \beta^2)$$

$$\kappa = \sqrt{3(1 - \Theta_c)}$$

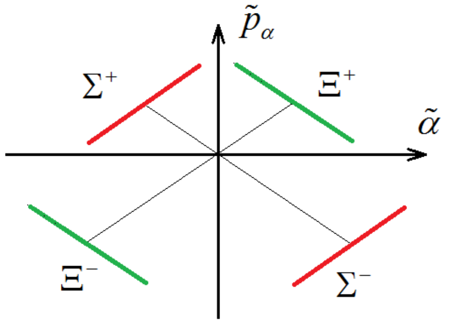
Construction of the local map (L-GR)

Auxiliary hyperplanes

$$\Sigma^{\pm} = \left\{ \tilde{\mathbf{z}} \mid \tilde{p}_\alpha - \kappa\tilde{\alpha} = \pm\delta_\pm \right\}$$

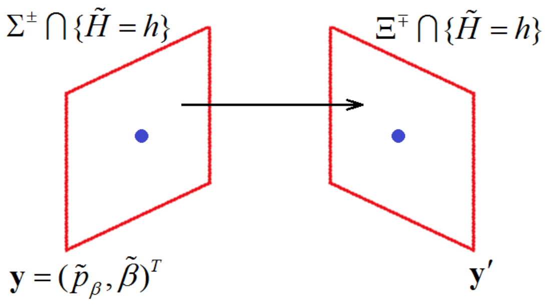
$$\Xi^{\pm} = \left\{ \tilde{\mathbf{z}} \mid \tilde{p}_\alpha + \kappa\tilde{\alpha} = \pm\delta_\pm \right\}$$

$$\delta_\pm > 0,\ \delta_+ \neq \delta_-$$

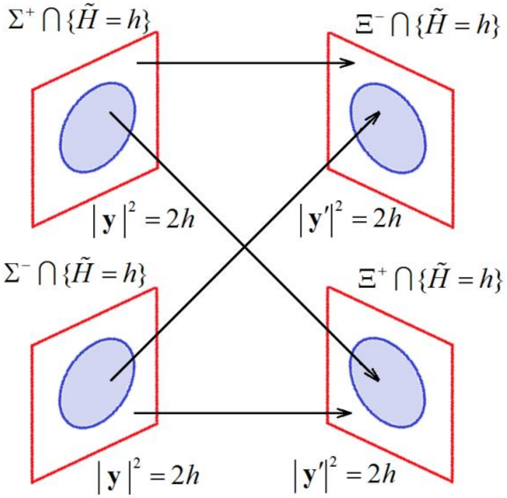


Local map:

$h \le 0$

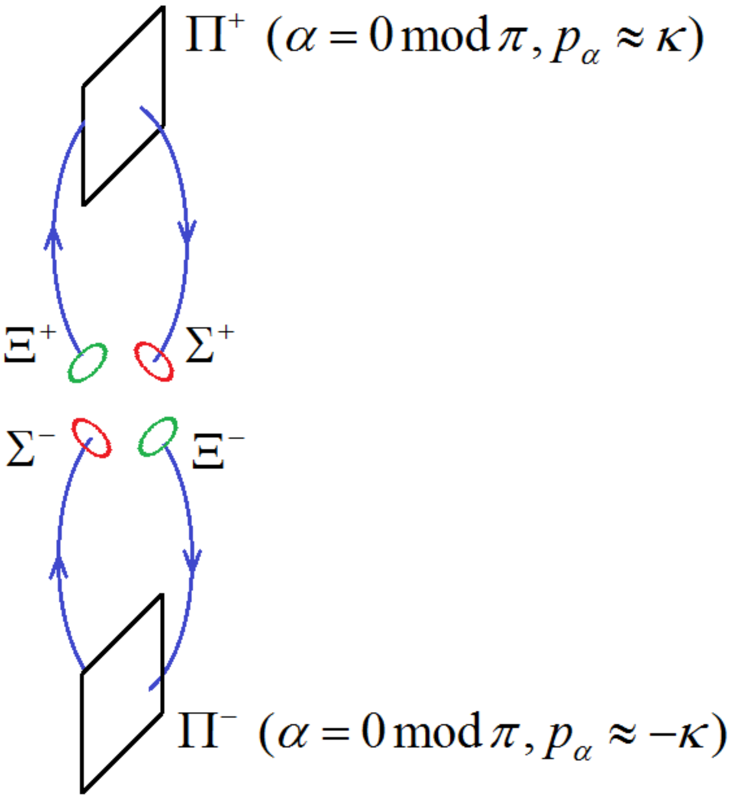


$h > 0$



$$\theta(\mathbf{y}) \approx -\frac{1}{\kappa}\ln\left| 2h - |\mathbf{y}|^2 \right| + c_0 \qquad c_0 = \frac{1}{\kappa}\ln \delta_+\delta_-$$

$$\theta(\mathbf{y}) \approx -\frac{1}{\kappa}\ln\left| 2h - |\mathbf{y}|^2 \right| + c_\pm \qquad c_\pm = \frac{2}{\kappa}\ln \delta_\pm$$



$$\Gamma_{\pm} : \Xi^{\pm} \to \Sigma^{\pm}$$

$$\Gamma_{+} = \begin{pmatrix} 0 & a \\ -\frac{1}{a} & 0 \end{pmatrix}, \quad \Gamma_{-} = \begin{pmatrix} 0 & b \\ -\frac{1}{b} & 0 \end{pmatrix}$$

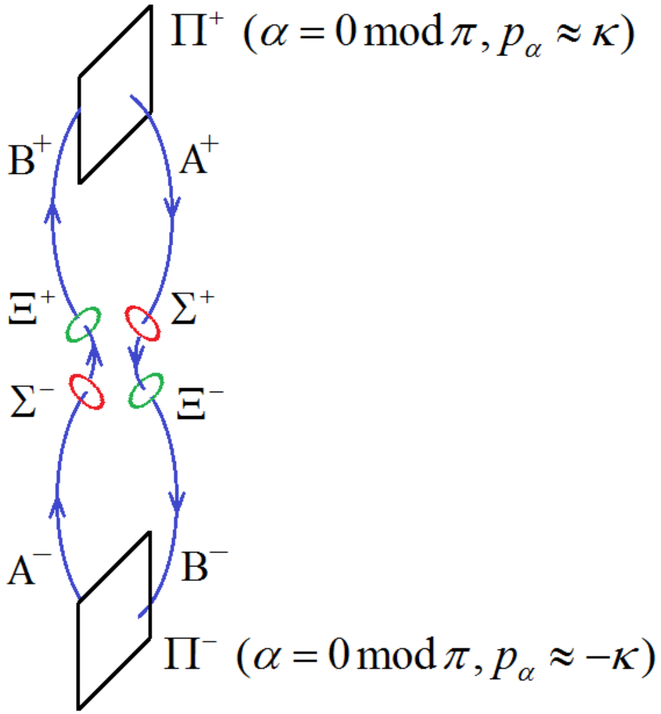
$$\Gamma_{\pm} = A_{\pm} B_{\pm}$$

$$A_{\pm} : \Pi^{\pm} \to \Sigma^{\pm}, \quad B_{\pm} : \Xi^{\pm} \to \Pi^{\pm}$$

$$A_{+} = \begin{pmatrix} \frac{\sigma_{+} a}{\sqrt{2}} & \frac{1}{\sigma_{+}\sqrt{2}} \\ -\frac{\sigma_{+}}{\sqrt{2}} & \frac{1}{a\sigma_{+}\sqrt{2}} \end{pmatrix} \qquad A_{-} = \begin{pmatrix} \frac{\sigma_{-} b}{\sqrt{2}} & \frac{1}{\sigma_{-}\sqrt{2}} \\ -\frac{\sigma_{-}}{\sqrt{2}} & \frac{1}{b\sigma_{-}\sqrt{2}} \end{pmatrix}$$

$$B_{\pm} = Q A_{\pm}^{-1} Q \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Composition of local and "global" maps



$$\text{Map } F : \Pi^+ \cup \Pi^- \to \Pi^+ \cup \Pi^-$$

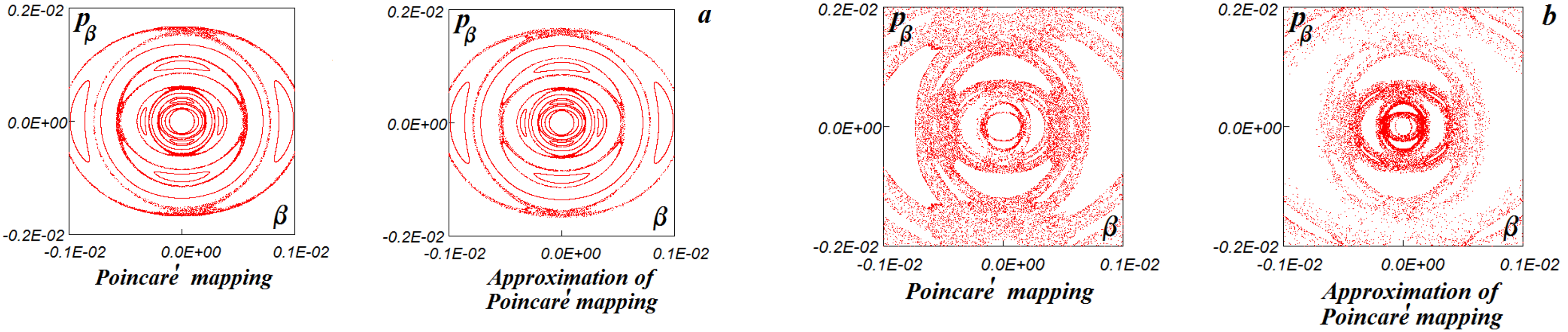
$$h = 0$$

$$F(\mathbf{y}) = \begin{cases} B_- R(c_0 - \frac{1}{\kappa} \ln |A_+ \mathbf{y}|^2) A_+ \mathbf{y}, & \mathbf{y} \in \Pi^+ \\ B_+ R(c_0 - \frac{1}{\kappa} \ln |A_- \mathbf{y}|^2) A_- \mathbf{y}, & \mathbf{y} \in \Pi^- \end{cases}$$

$$F(\Pi^\pm) = \Pi^\mp, \quad F^2(\Pi^\pm) = \Pi^\pm$$

Comparison with Poincare mapping

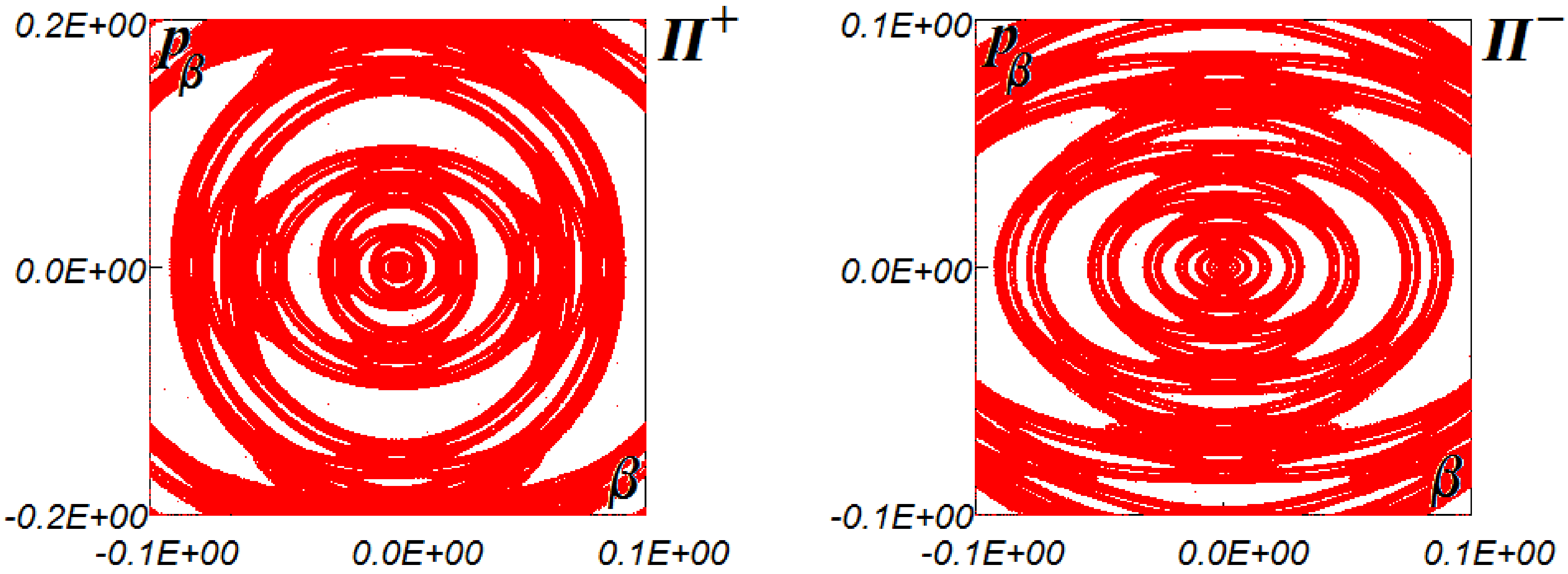
$$h=0,\ F^2 : \Pi^+ \to \Pi^+$$



Section of the plane $\alpha = 0(\mathrm{mod}\,\pi)$ by trajectories in which the angle α increases and its approximation by the mapping $F^2 : \Pi^+ \to \Pi^+$. The value of the Hamiltonian $h = 0$. In the case a the ratio of the moments of inertia of the satellite $\theta_c = 0.87$; numerical study indicates that the mapping has invariant curves enclosing the point $\mathbf{y} = 0$. In the case b the parameter $\theta_c = 0.82$ and there are no invariant curves enclosing $\mathbf{y} = 0$.

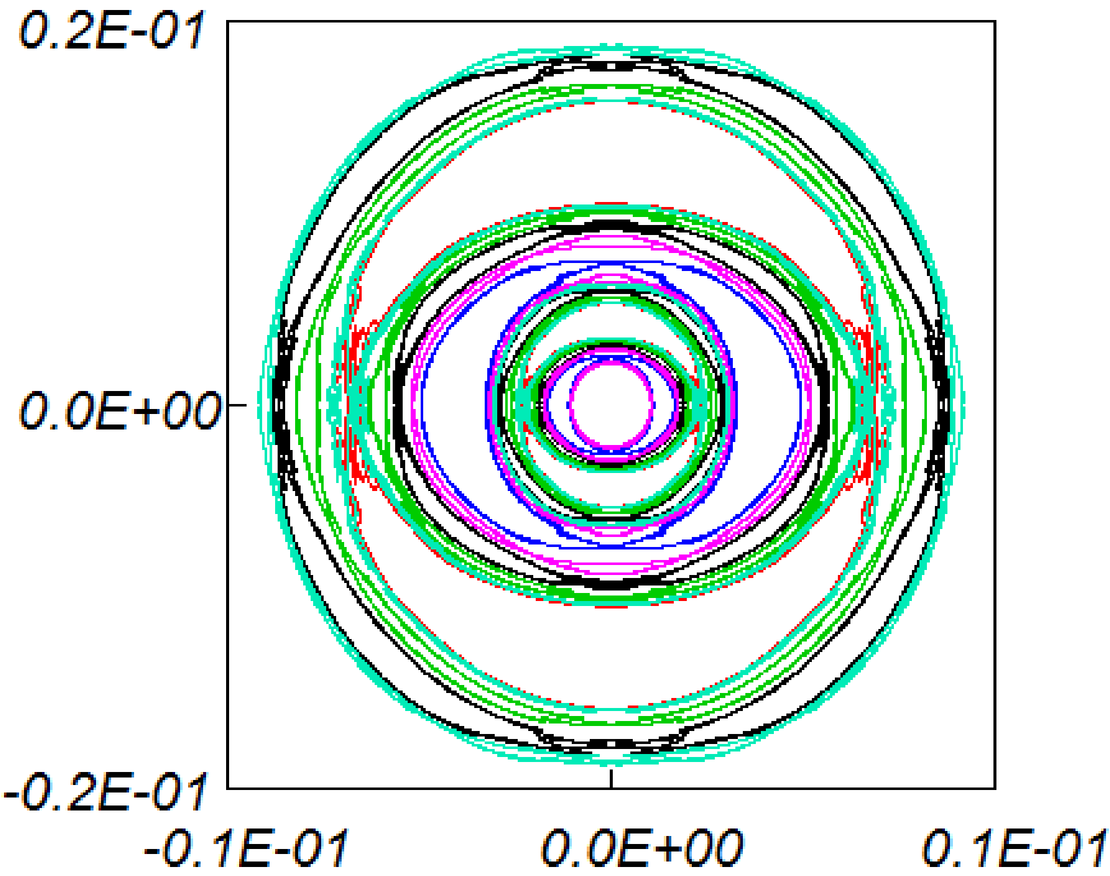
Scaling (h=0)

$$F\left(e^{\frac{\kappa n\pi}{2}}\tilde{\mathbf{y}}\right) = (-1)^n e^{\frac{\kappa n\pi}{2}} F(\tilde{\mathbf{y}}),\ n \in Z$$



$$\theta_c = 0.85$$

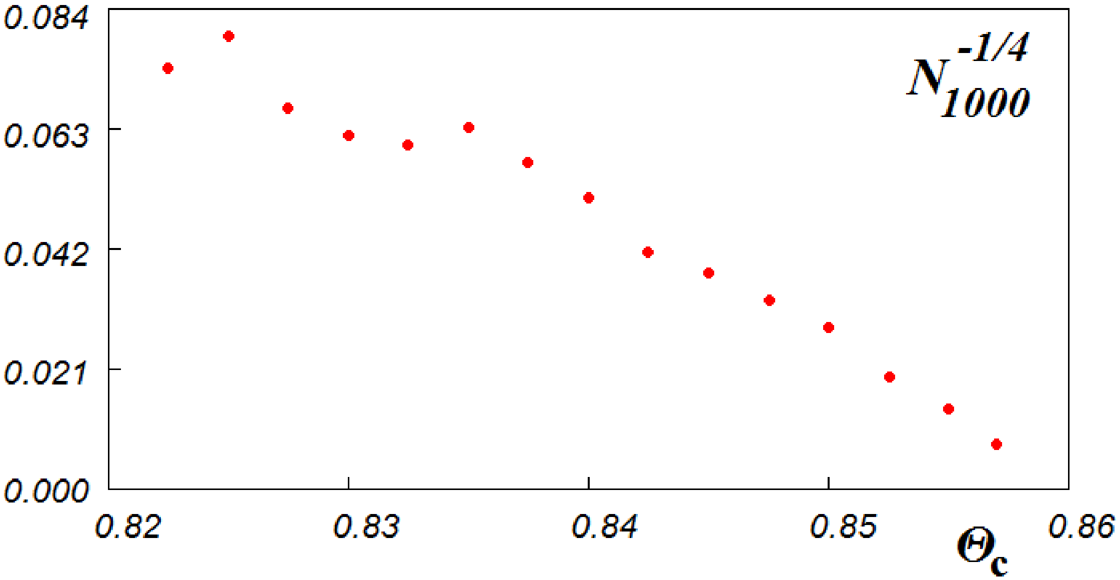
Fixed points of F^{2n} (h=0)



$$\theta_c = 0.84$$

Stability of a separatrix contour (h=0)

Empirical approach: 30 points on a semicircle with a radius of 0.001, N_{1000} is the minimum number of iterations required to move one of the points to a unit distance from the origin



Factorization (h=0)

$$\tilde{\mathbf{y}}_{\Sigma^\pm} = \sqrt{2J_{\Sigma^\pm}} \begin{pmatrix} \cos\left(\varphi_{\Sigma^\pm} + \frac{1}{\kappa} \ln J_{\Sigma^\pm}\right) \\ \sin\left(\varphi_{\Sigma^\pm} + \frac{1}{\kappa} \ln J_{\Sigma^\pm}\right) \end{pmatrix}$$

Modified polar symplectic coordinates

Formulas for mapping in polar coordinates:

$$J_{\Sigma^-} = J_{\Sigma^+} \sqrt{\beta^2 \sin^2 \varphi_{\Sigma^+} + \frac{1}{\beta^2} \cos^2 \varphi_{\Sigma^+}}$$

$$\varphi_{\Sigma^-} = \left[\operatorname{arctg}\left(\beta^2 \operatorname{tg} \varphi_{\Sigma^+}\right) - \frac{1}{\kappa} \ln J_{\Sigma^-} \right] (\operatorname{mod} \pi)$$

Factorization (h=0)

Factoring change of variables:

$$J_{\Sigma^{\pm}} \rightarrow \eta_{\Sigma^{\pm}} = -\frac{1}{\kappa}\ln J_{\Sigma^{\pm}}$$

Factorized mappings:

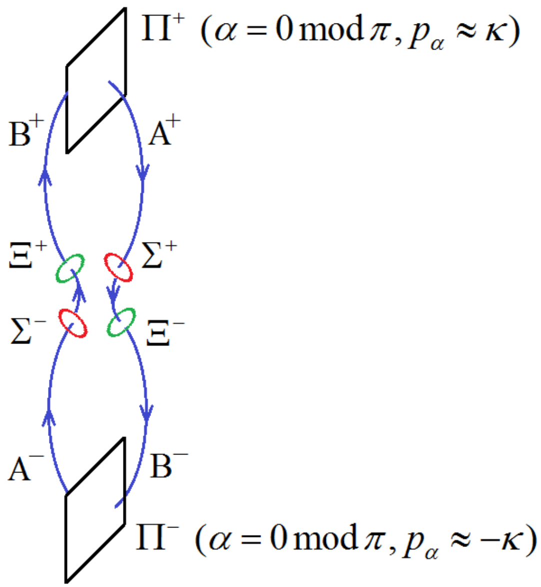
$$\eta_{\Sigma^-} = \left[\eta_{\Sigma^+} - \frac{1}{\kappa}\ln\sqrt{\beta^2\sin^2\varphi_{\Sigma^+} + \frac{1}{\beta^2}\cos^2\varphi_{\Sigma^+}}\right](\bmod\,\pi)$$

$$\varphi_{\Sigma^-} = \left[\operatorname{arctg}\left(\beta^2\operatorname{tg}\varphi_{\Sigma^+}\right) + \eta_{\Sigma^-}\right](\bmod\,\pi)$$

Preserved measure

$$e^{-\kappa\eta_{\Sigma^{\pm}}}\,d\eta_{\Sigma^{\pm}}\,d\varphi_{\Sigma^{\pm}}$$

Motion properties at h<0



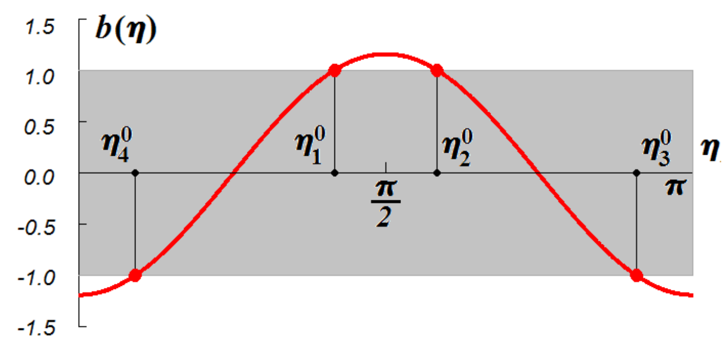
Stability of fixed point y=0

$$b_0(\eta) = \frac{1}{2} tr\left(D\left(F_{\Sigma^-} \circ F_{\Sigma^+}\right)\Big|_{\mathbf{y}_{\Sigma^-}=0} \right) =$$

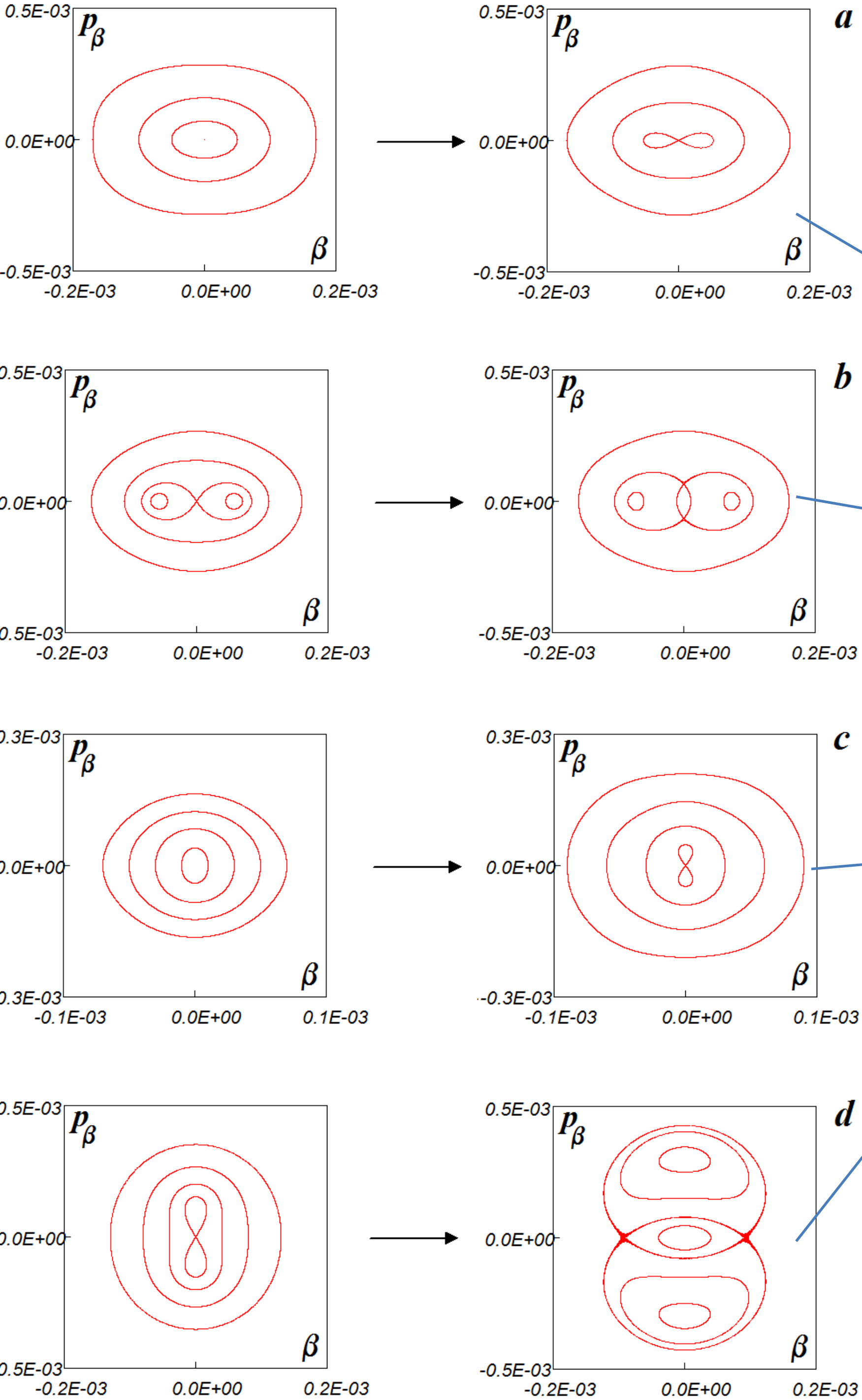
$$= \frac{1}{4}\left[\left(\alpha - \frac{1}{\alpha}\right)\left(\beta - \frac{1}{\beta}\right) - \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)\cos 2\eta\right].$$

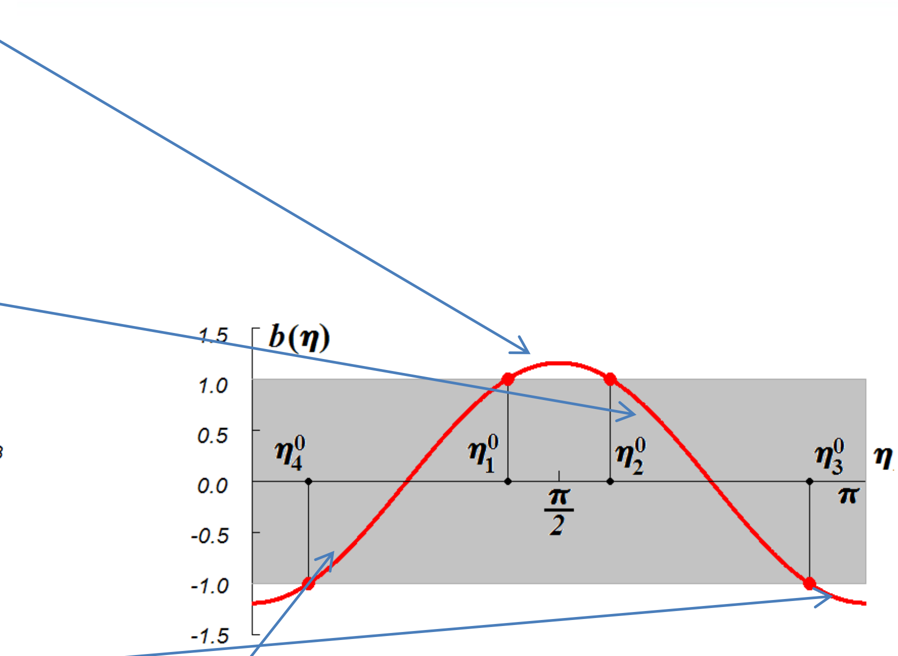
$$\eta = -\frac{1}{\kappa}\ln|2h| + c_0$$

The alternation of stability and instability in the family of plane oscillations

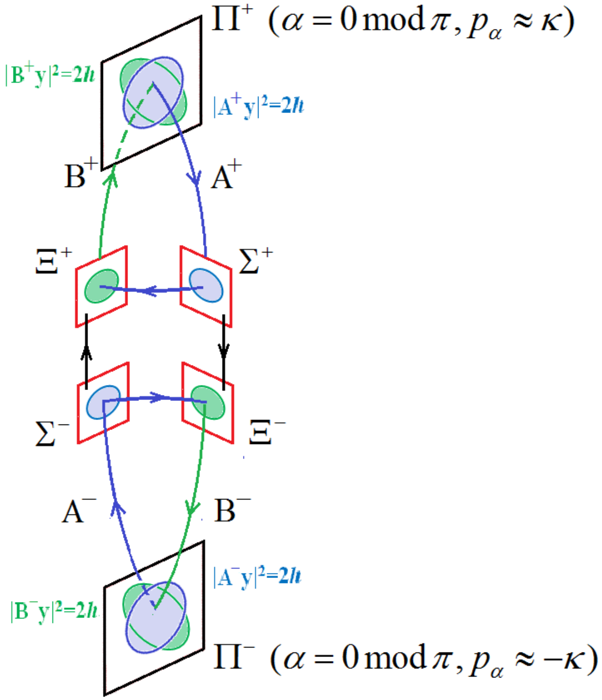


Motion properties at h<0





Motion properties at h>0



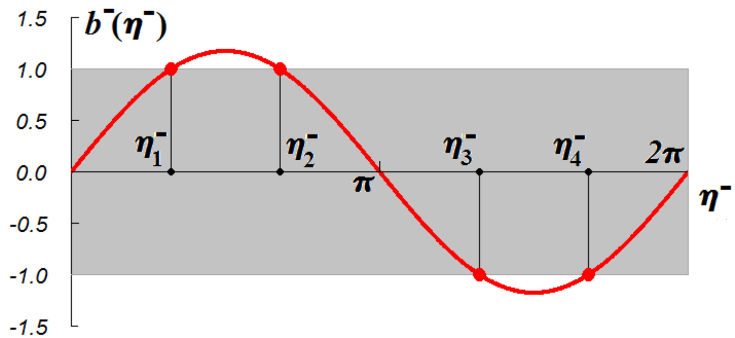
The alternation of stability and instability in the family of planar rotations

Stability of fixed point y=0

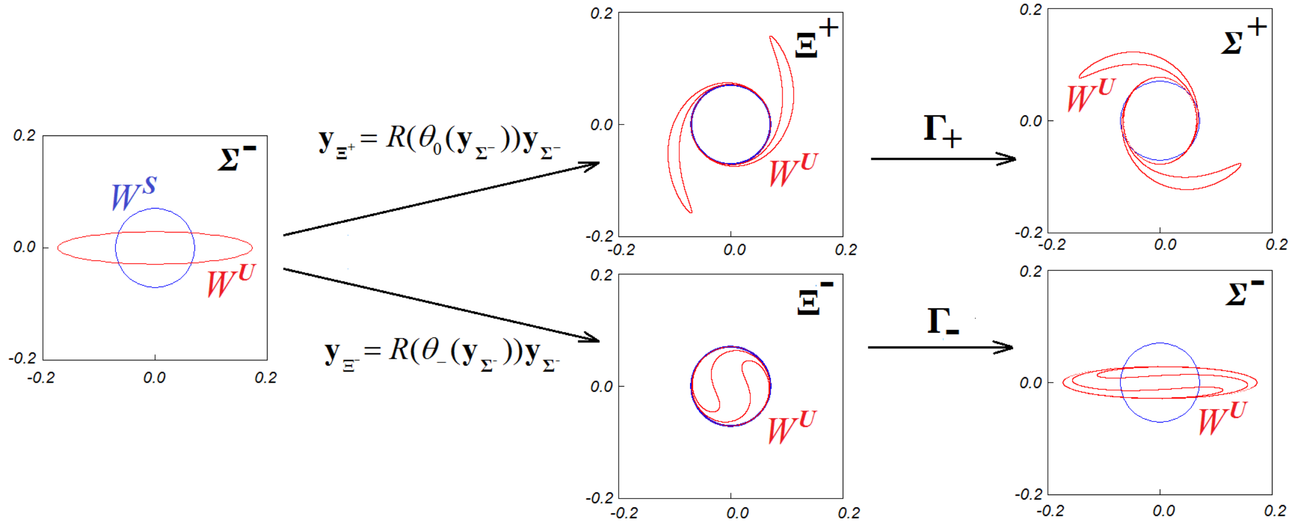
$$b^+(\eta^+) = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)\sin\eta^+$$

$$b^-(\eta^-) = \frac{1}{2}\left(\beta + \frac{1}{\beta}\right)\sin\eta^-$$

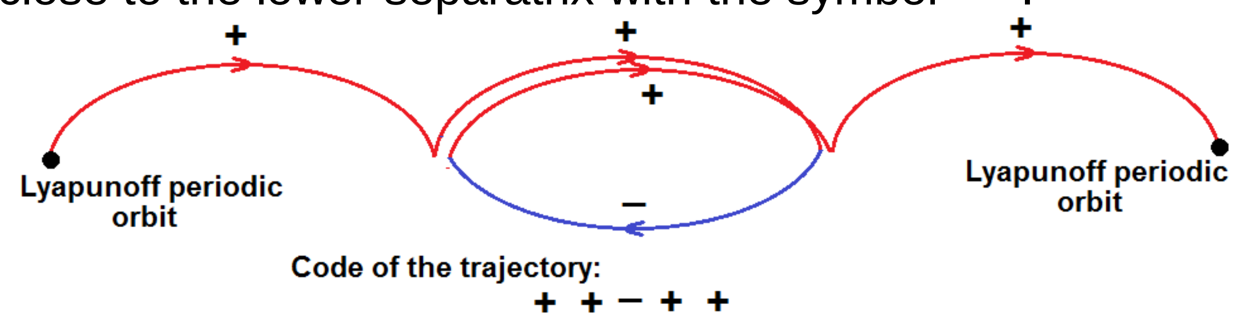
$$\eta^\pm = -\frac{1}{\kappa}\ln(2h) + c^\pm$$



Coding of doubly asymptotic motions (h>0)



Let's mark the motion close to the upper separatrix with the symbol ``+'' and mark the motion close to the lower separatrix with the symbol ``-''.

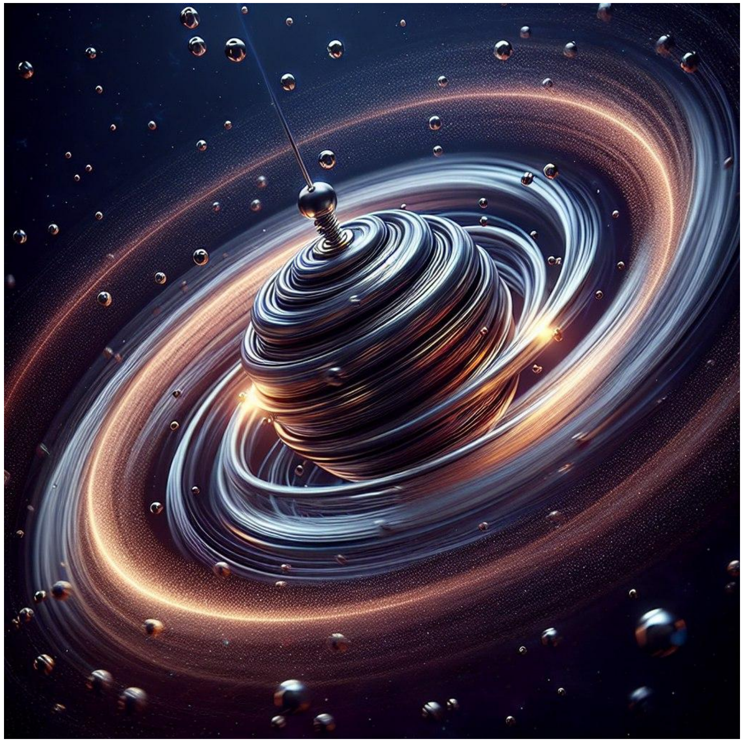


Proposition: there are possible doubly asymptotic motions corresponding to any finite sequence of symbols + and -.

Conclusion & Future Work

- The analysis revealed previously unknown properties of the rotational motion of a body in a gravitational field
- Approximate formulas for the mappings generated by the phase flow simplify numerical studies
- It would be interesting to consider the case of an asymmetric body or a case where the projection of the kinetic moment on the axis of symmetry of the body is nonzero.

Picture generated by a neural network for the request
“rotational motion of a rigid body in a gravitational field”



Thank you for your attention!