Localized trajectories of cosmic particles near libration points

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Abstract. We discuss a new behavior of a dynamic system near an unstable equilibrium position. We call corresponding trajectories located in the selected neighborhood of unstable equilibrium as localized trajectories.

The use of topological methods for proving the existence of localized trajectories makes possible to abandon the condition of analyticity of the first integrals and the condition of non-resonance of purely imaginary roots of the characteristic equation for the systems of Lyapunov.

As an important application, we consider perturbed motion in libration points vicinity of the restricted circular three-body problem. Numerical simulations for the parameters of the Earth-Moon system convincingly illustrate our theoretical study.

Introduction

Let us consider a dynamical system whose equilibrium position is non-degenerate and unstable in Lyapunov sense, and its degree of instability is greater than zero and less than the number of degrees of freedom. When considering the behavior of a mechanical system near an equilibrium position or near a steady state of motion, when higher-order terms in the expansions of kinetic and potential energies are also taken into account, one has a system of differential equations with additional non-linear terms. A.M. Lyapunov showed that, under very general assumptions, such a system admits periodic solutions of a certain type and indicated an effective way to calculate these solutions.

Our work deals with a situation where a mechanical system with $n > 1$ degrees of freedom has a non-degenerate Lyapunov unstable equilibrium position, the degree of instability of which ν lies within $1 \leq \nu \leq n-1$. The energy at the equilibrium position is assumed to be zero. It is shown that for any sufficiently small positive value of the total energy of the system, there is a motion of the system with a given energy value that begins at the boundary of the region where motion is possible and does not leave a small neighborhood of the equilibrium position. We call such motions as localized motions.

An essential condition for the presence of such movements is the limitation of system movements in "unstable directions." For natural systems with gyroscopic and dissipative forces, this is ensured by the conservation or non-increase of the total mechanical energy. The use of topological Ważewski method applying the Borsuk concept of retract [1, 2] in the analysis of such motions makes possible to abandon the condition of analyticity of the first integrals and the condition of non-resonance of purely imaginary roots of the characteristic equation. The presence of time-dependent gyroscopic and dissipative forces, as well as forces with incomplete dissipation, does not interfere with the proof of the existence localized solutions [3].

As an example, we consider the planar restricted circular three-body problem. Two triangular libration points have an even degree of instability. For certain mass ratios of the two main bodies they are gyroscopically stable, and we don't consider them in our application.

Three collinear libration points have degree of instability equal to unity, therefore, according to the Kelvin-Chetaev theorem, they cannot be stabilized by adding dissipative and gyroscopic forces. Nevertheless, in accordance with our research, localized trajectories should exist near these unstable collinear libration points. Numerical simulations for the parameters of the Earth-Moon system convincingly illustrate our theoretical study.

1. Perturbed linear system of the second order.

We consider the following system:

$$\ddot{x} = A(t)x + B(t)\dot{x} + \mu g(x, \dot{x}, t), \qquad x \in W \subseteq R^n, \qquad t \geq 0; \tag{1}$$

where $\mu \geq 0$ — parameter, W — an open domain containing point $x = 0$.

It is assumed that in the domain W for $t \geq 0$ the matrices $A(t), B(t)$ and the function $g(x, t)$ are continuous in (x, t) and norm-bounded:

$$\|A(t)\| \leq a, \qquad \|B(t)\| \leq b, \qquad \|g(x,t)\| \leq d, \qquad x \in W, \qquad t \geq 0, \tag{2}$$

for some constants a, b, d.

Definition For $\varepsilon > 0$ we introduce an open neighborhood $U_\varepsilon = \{x : \|x\| < \varepsilon, x \in R^n\}$. A solution $x(t)$ of system (1) will be called localized in a neighborhood of U_ε, if it begins at $t = 0$ in this neighborhood, exists for $t \geq 0$, and does not leave U_ε for $t \geq 0$.

Theorem.

Let the matrix A be symmetric for $t \geq 0$, its characteristic numbers are positive, bounded, and separated from zero uniformly in t, i.e.

$$c\|x\|^2 \leq (A(t)x, x) \leq a\|x\|^2, \qquad \text{for} \qquad t \geq 0, \qquad \forall x \in R^n \tag{3}$$

where $c > 0$ — some constant, and

$$c - b\sqrt{a} > 0.$$

Let also the matrix $\dot{A}$ be non-negative definite, and the matrix $B(t)$ be non-positive definite, and the perturbation $g(x, \dot{x}, t)$ is dissipative, for all $x \in U_\varepsilon$, $t \geq 0$:

$$(\dot{A}(t)\dot{x}, \dot{x}) \geq 0, \qquad (B(t)\dot{x}, \dot{x}) \leq 0, \qquad (g(x, \dot{x}, t), \dot{x}) \leq 0, \qquad \forall \dot{x} \in R^n. \tag{4}$$

And let for $t \geq 0$ the function $g(x, t)$ satisfy the Lipschitz condition uniformly in t, i.e. there is $L > 0$ such that

$$\|g(x_1, t) - g(x_2, t)\| \leq L\|x_1 - x_2\|, \qquad \text{for} \qquad t \geq 0. \tag{5}$$

Let us choose an arbitrary $\varepsilon > 0$ such that $U_\varepsilon \subseteq W$. Then there is $\mu^0 > 0$ that for all values of the parameter μ such that $0 \leq \mu < \mu^0$, there exists a solution of system (1), localized in a neighborhood of U_ε.

2. Collinear libration points

Lagrange equations of perturbed planar restricted three-body problem read:

$$\begin{array}{l} \ddot{x}_1 = -\omega^2 x_1 + c\dot{x}_2 + f_1(\mathbf{x}, \dot{\mathbf{x}}) \\ \ddot{x}_2 = \quad \alpha^2 x_2 - c\dot{x}_1 + f_2(\mathbf{x}, \dot{\mathbf{x}}), \end{array} \qquad f_i = O(\mathbf{x}^2 + \dot{\mathbf{x}}^2), \qquad i = 1, 2, \tag{6}$$

where c — some constant (possibly, $c(t)$), intensity of gyroscopic forces.

We fix $h > 0$. Area of possible of motion of unperturbed problem is the following: $\omega^2 x_1^2 - \alpha^2 x_2^2 \leq 2h$. Let us define a closed subdomain W in it: $\alpha^2 x_2^2 \leq 4h$. Figure 1 shows the trajectories starting with zero-velocity (acceleration is greater than zero) from the left part of the boundary of W. Libration point $L1$ is in the center of W — point $(0, 0)$. The upper part of the trajectories leave the vicinity of the libration point through the upper part of the boundary, the lower part of the trajectories leave through the lower part of the border. It means, that for each energy level at least one trajectory starting on the left side of the boundary will remain in the vicinity of the libration point, which is proven in Theorem.

Conclusion

For natural systems with two degrees of freedom, localized motions are periodic, similar to the result of the corresponding theorem for Lyapunov systems. In the general case, our proof does not require the conditions of non-resonance of purely imaginary roots of the characteristic equation and the presence of an analytical first integral of the dynamical system. With addition of gyroscopic forces and of dissipative forces (with or without complete dissipation), and possibly time-dependent ones, the existence of localized motions is also proven.

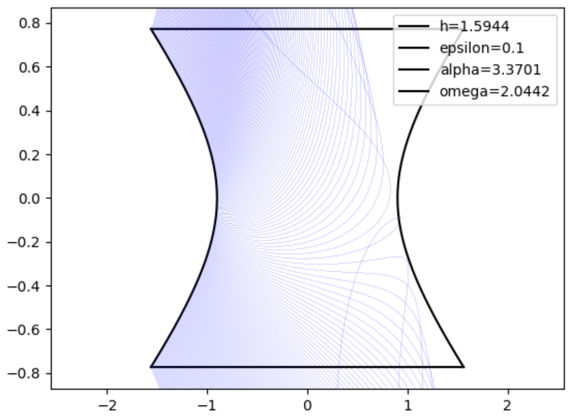

FIGURE 1. L1 vicinity

References

[1] K.Borsuk, *Theory of retracts* , Polish Scientific Publishers, 1967.

[2] D.Reissig, G.Sansone, and R.Conti, *Qualitative theorie nichtlinearer differentialgleichungen*, Roma: Edizioni Cremonese, 1963.

[3] E.I.Kugushev, T.V.Salnikova *Existence of localized motions in the vicinity of an unstable equilibrium position* , Proceedings of the Steklov Mathematical Institute. 2024, V. 4.

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