International Conference "Analytical Methods of Celestial Mechanics"

Localized Trajectories of Cosmic Particles near Libration Points

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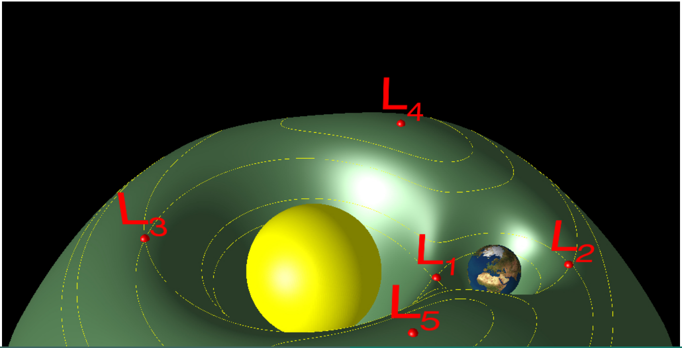
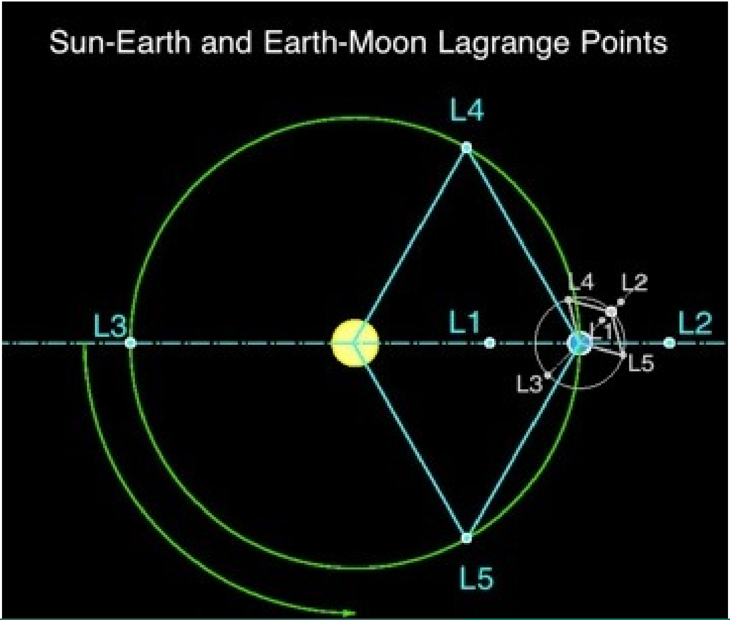


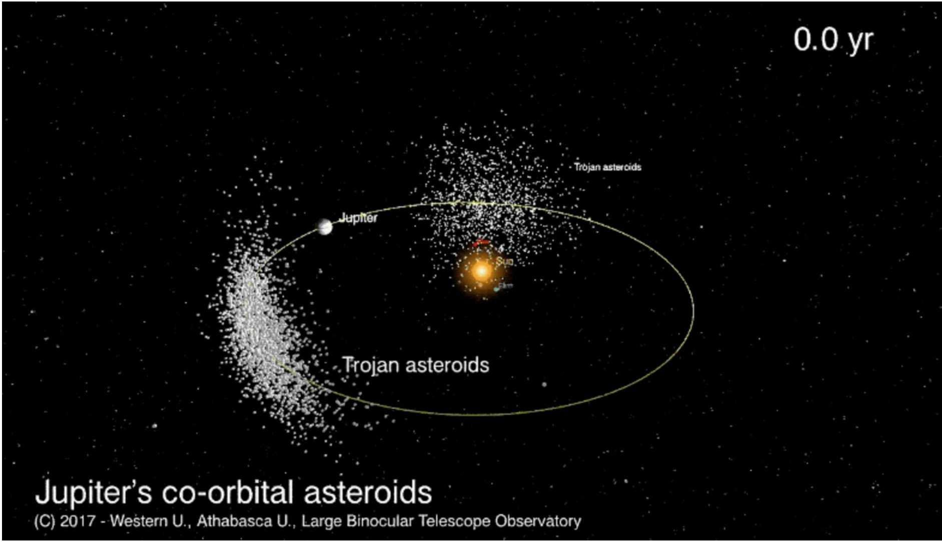
On the stability of motion of dynamical systems

- Definition of Lyapunov stability; Lyapunov theorem on stability; Lyapunov's theorems on asymptotic stability and instability on linear approximation.
- Lagrange-Dirichlet theorem for a natural mechanical system; Kelvin-Chetaev theorems on the influence of gyroscopic and dissipative forces on the stability of the equilibrium position (even/odd degree of instability), steady states.
- Theorem on the existence of periodic solutions in Lyapunov systems (analytical first integral, autonomy and non-resonance required). (close to even unstable equilibrium position)

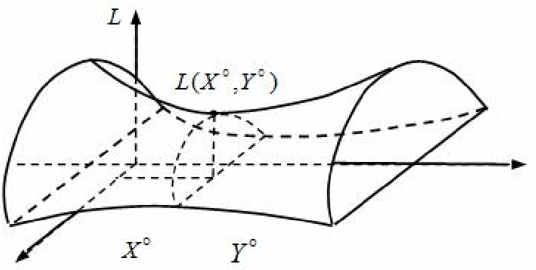
Stable-unstable solutions



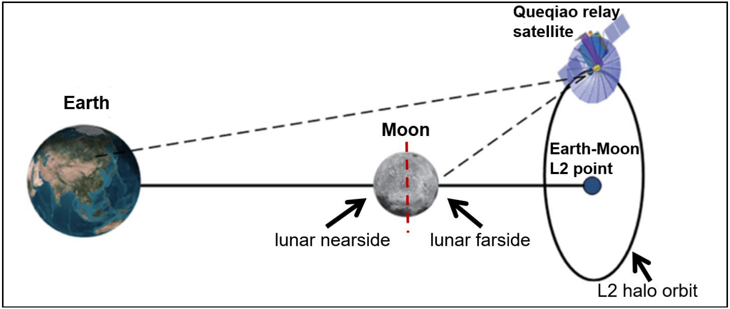
Gyroscopic stabilization



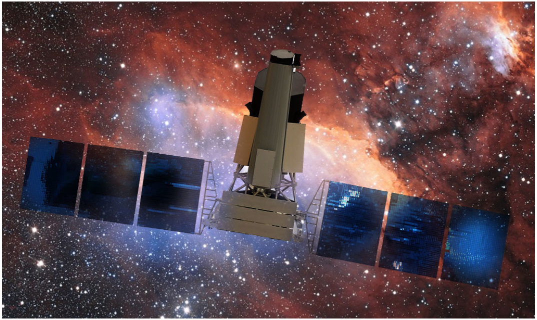
Existence of periodic orbits in Lyapunov systems

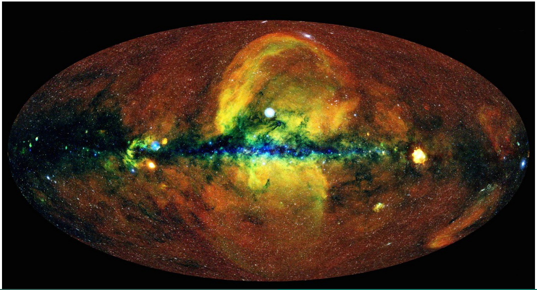


Existence of periodic orbits



Space project "Spektr-RG"(Sun-Earth L_2 point)





PROBLEM SETTING

- For nonlinear systems there are various behavior scenarios - orbital stability, Poisson stability, Poincaré and Birkhoff stability, Laplace (Jacobi) stability, conditional stability and so on.
- We discuss a new behavior of a dynamic system near an unstable equilibrium position - let's call it localized movements. These are trajectories located in the selected neighborhood of unstable equilibrium.
- When proving the existence of localized trajectories, we do not require such conditions as autonomy, absence of resonances, presence of an analytical first integral.

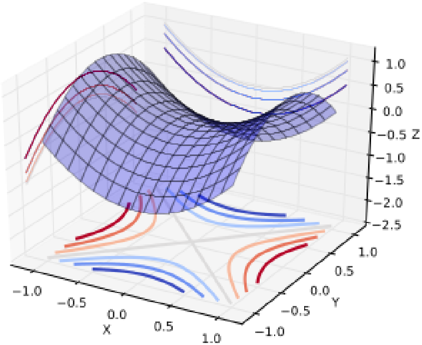
DEFINITION

- If in an equilibrium position the potential energy of a natural system reaches a strict local minimum, then, by virtue of Bolotin's theorem, at small positive values of the total energy there is at least one periodic solution.
  In this case, the equilibrium position is stable in Lyapunov sense.
- We consider a situation where the equilibrium position is non-degenerate and Lyapunov unstable. Moreover, its degree of instability is not zero and is less than the number of degrees of freedom.
- We prove that for any sufficiently small positive value of the total energy, there is a motion with a given energy value that does not leave a small neighborhood of the equilibrium position. These motions we call LOCALIZED MOTIONS.

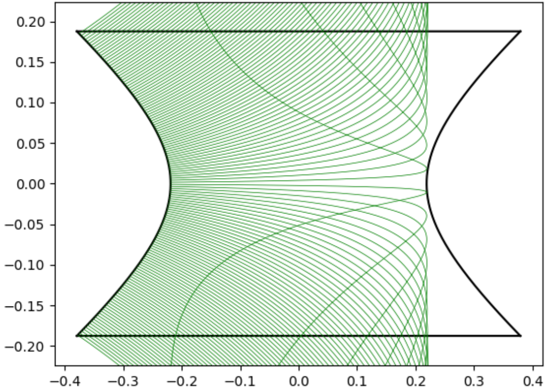
LOCALISED MOTIONS

- If the equilibrium position is stable in Lyapunov sense, all solutions in it's vicinity can be called LOCALIZED, among them there are periodic solutions.
- If the equilibrium position is unstable, and system is of Lyapunov type, the corresponding Lyapunov periodic solution can be called LOCALIZED.
- The goal of current study is to show in more general case the existence of LOCALIZED MOTIONS.

Example of potential energy with saddle point

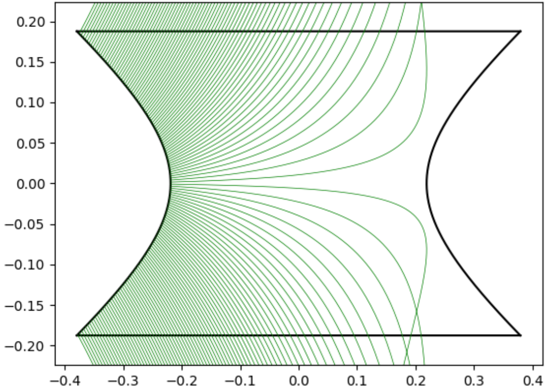


Example



$$\ddot{x}_1 = -x_1$$
$$\ddot{x}_2 = x_2 \qquad (1)$$

Example



$$\begin{aligned} \ddot{x}_1 &= -\omega^2 x_1 \\ \ddot{x}_2 &= \ \ \alpha^2 x_2 \end{aligned} \tag{2}$$

Lagrangian systems — natural system

THEOREM 1.
$\mathrm{q} = (q_1, \ldots, q_n)$

$$L(\mathrm{q}, \dot{\mathrm{q}}) = \frac{1}{2}\left(A(\mathrm{q})\dot{\mathrm{q}}, \dot{\mathrm{q}}\right) - V(\mathrm{q}).$$

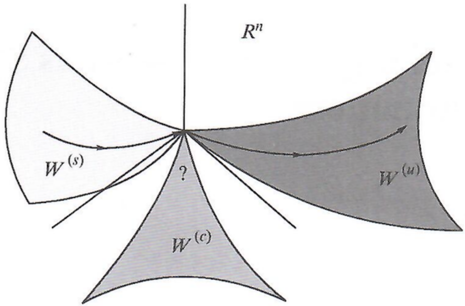
$V(\mathrm{q})$ and $A(\mathrm{q}) \in C^2[\mathrm{q}]$.
Let $\mathrm{q} = 0$ - non-degenerate and Lyapunov unstable equilibrium position, degree of instability η, $1 \leq \eta \leq n-1$. Then for any sufficiently small $\varepsilon > 0$ it exists $h_0 > 0$, for any h: $0 \leq h \leq h_0$ il exist $\mathrm{q}(t)$, with energy h, $q_1^2 + \ldots + q_n^2 < \varepsilon$.

Remark

Unlike Lyapunov theorem on the existence of periodic solutions in Lyapunov systems our result remains in the case of resonances for ω_s.

Stable, unstable and central manifolds

Fig. 1.1 Stable, center and unstable manifolds



Localized solutions belong to the central manifold.

For non-autonomous systems, invariant manifolds disappear —>

Lagrangian systems — system with gyroscopic and dissipative forces

THEOREM 2.

Let $q = 0$ - non-degenerate and Lyapunov unstable equilibrium position, degree of instability η, $1 \leq \eta \leq n-1$.

Let gyroscopic and dissipative forces — generalized forces of the form

$Q = C(t)\dot{q} + O(q^2 + \dot{q}^2)$,

where C — either a constant or non-constant $n \times n$ matrix such that $\|C\| \leq \psi$, where ψ — some positive small quantity determined by the parameters of the system.

Then for any sufficiently small $\varepsilon > 0$

it exists $h_0 > 0$, such that for any h: $0 \leq h \leq h_0$ it exists (localized) solution $q(t)$, with initial energy h, always in the area $q_1^2 + \ldots + q_n^2 < \varepsilon$.

Particular case — system with gyroscopic forces, $n = 2$

THEOREM 3.

Let q $= 0$ - non-degenerate and Lyapunov unstable equilibrium position, its degree of instability $\eta = 1$.
Let gyroscopic forces are small.
Then for any sufficiently small $\varepsilon > 0$
it exists $h_0 > 0$, such that for any h: $0 \leq h \leq h_0$
it exists solution q(t), with initial energy h, always in the area $q_1^2 + q_2^2 < \varepsilon$.

Comments

In accordance with Morse Lemma, in some neighborhood of a non-degenerate critical point there is a local coordinate system in which $V(\mathrm{q}) = \frac{1}{2}\left(q_1^2 - q_2^2\right)$.

Using linear substitution we move to normal coordinates $\mathrm{x} = (x_1, x_2)$ in which $V(\mathrm{x}) = \frac{1}{2}\left(\omega^2 x_1^2 - \alpha^2 x_2^2\right)$, $\omega > 0$, $\alpha > 0$. With gyroscopic forces Lagrange equations read:

$$\begin{aligned} \ddot{x}_1 &= -\omega^2 x_1 + c\dot{x}_2 + f_1(\mathrm{x}, \dot{\mathrm{x}}) \\ \ddot{x}_2 &= \ \ \alpha^2 x_2 - c\dot{x}_1 + f_2(\mathrm{x}, \dot{\mathrm{x}}), \end{aligned} \qquad f_i = O(\mathrm{x}^2 + \dot{\mathrm{x}}^2), \qquad i = 1, 2, \qquad (3)$$

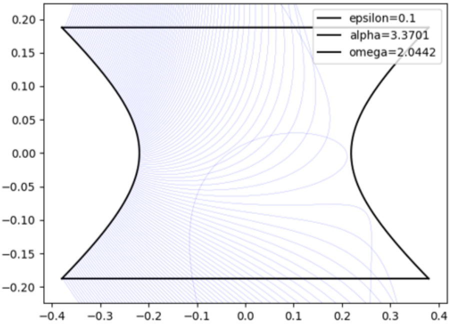
where c — some constant (possibly, $c(t)$)

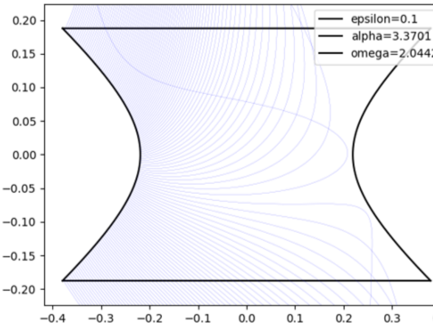
Comments

We fix $h > 0$.
Area of possible motion: $\omega^2 x_1^2 - \alpha^2 x_2^2 \leq 2h$. Let us define a closed subdomain W in it: $\alpha^2 x_2^2 \leq 4h$.
In W we have $\omega^2 x_1^2 \leq 6h$, and $\omega^2 x_1^2 + \alpha^2 x_2^2 \leq 10h$.

Boundary W consists of four parts: $\Gamma_{1,2}$ — $\{\omega x_1 = \mp\sqrt{2h + \alpha^2 x_2^2}$, $\alpha^2 x_2^2 \leq 4h\}$, on the boundary of the area of possible of motion;
and $\Gamma_{3,4}$ — $\{\alpha x_2 = \pm\sqrt{4h}$, $\omega^2 x_1^2 \leq 6h\}$.
To prove the theorem, the topological Waz̶ewski method applying the concept of retract of Borsuk is used.

Numerical simulation L1 — autonomous / non-autonomous cases





Remarks

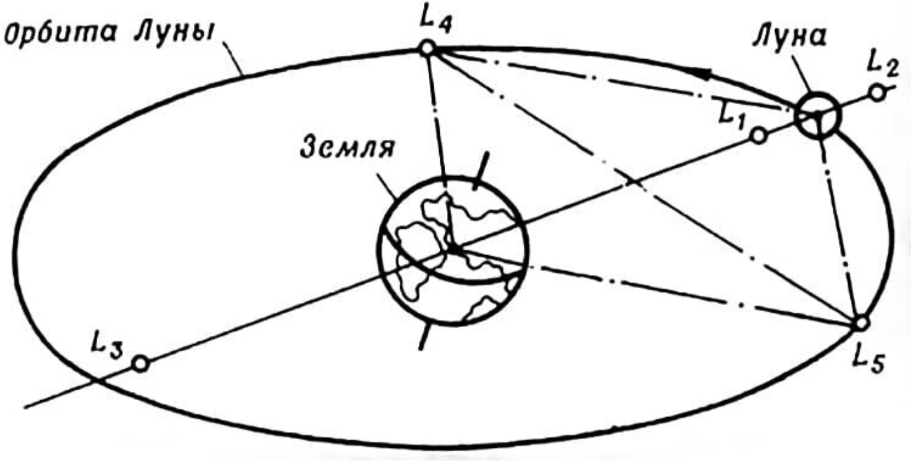
1. Theorem 3 gives us sufficient condition for existence of the localized motions,
$\alpha - |c|\sqrt{2} > 0$ is important.

2. It is possible to consider $c(t) < const$ (gyroscopic forces depending on time)

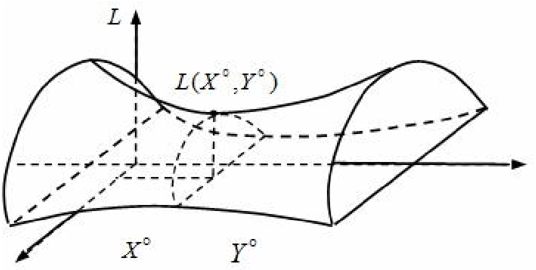
3. Adding dissipative forces does not contradict the proof of the theorem.

Е. И. Кугушев, Т. В. Сальникова. Существование локализованных движений в окрестности неустойчивого положения равновесия, Труды Математического института имени В. А. Стеклова, том 327.

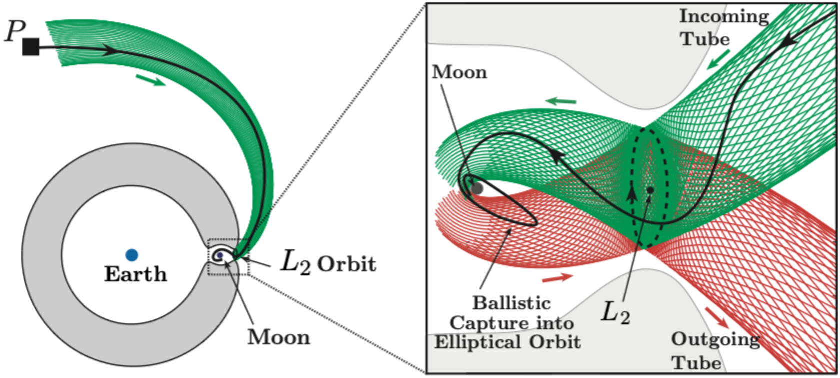
Libration points of the EARTH - MOON system



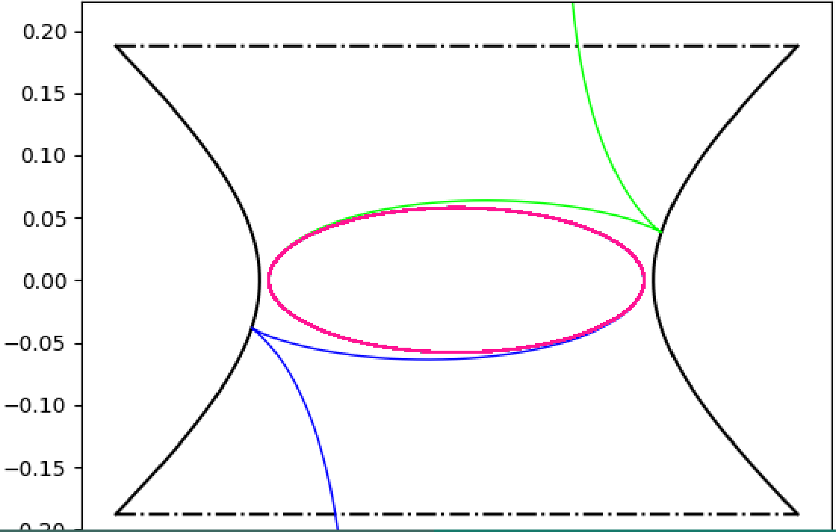
Existence of periodic orbits in Lyapunov systems



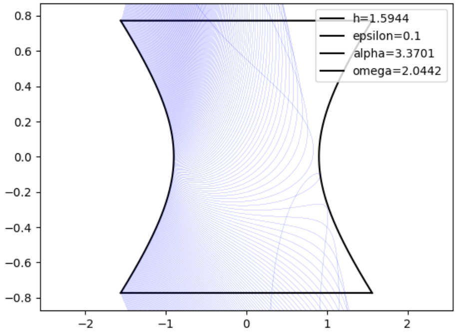
Invariant manifold structures of L1 and L2 (Marsden)



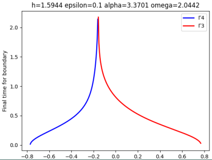
Localized solution (L1)



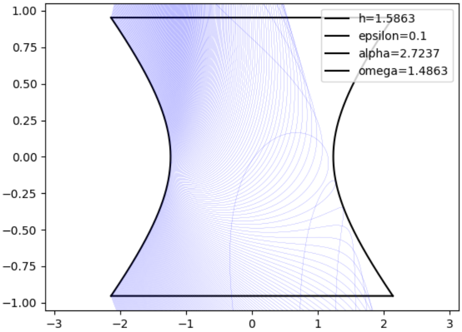
Numerical simulation - Libration point L1



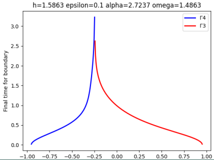
Sufficient condition for existence of the localized motions: $\alpha - |c|\sqrt{2} > 0$



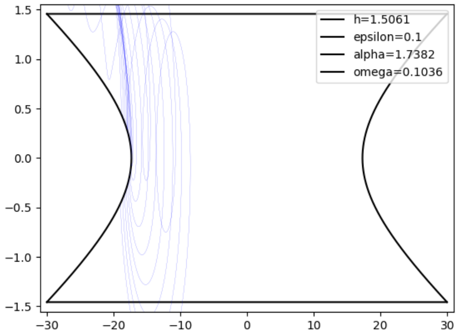
Numerical simulation - Libration point L2

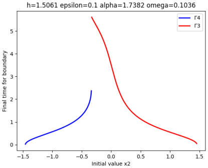


Sufficient condition for existence of the localized motions: $\alpha - |c|\sqrt{2} > 0$



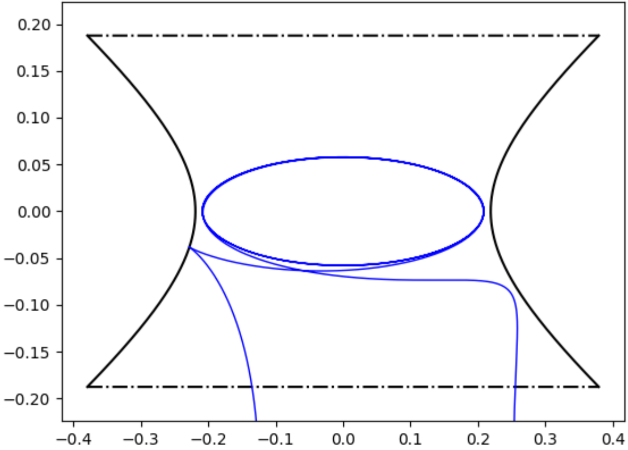
Numerical simulation - Libration point L3



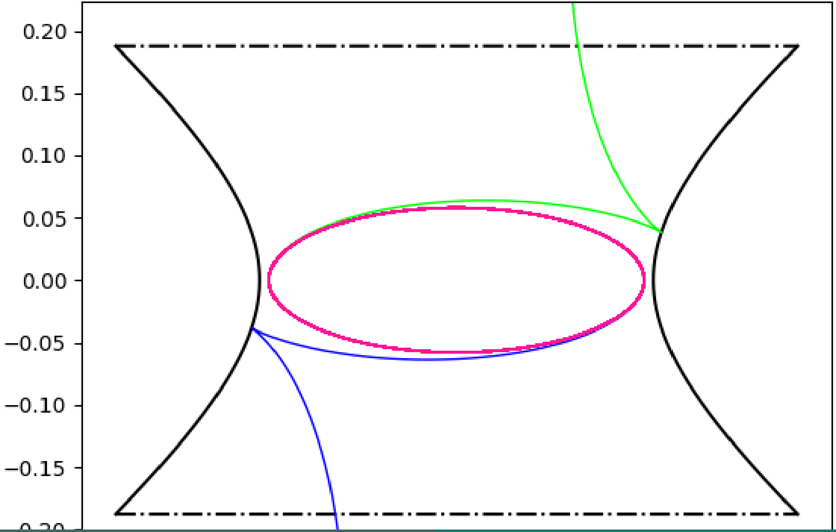


$$\alpha - |c|\sqrt{2} > 0 \ (?)$$

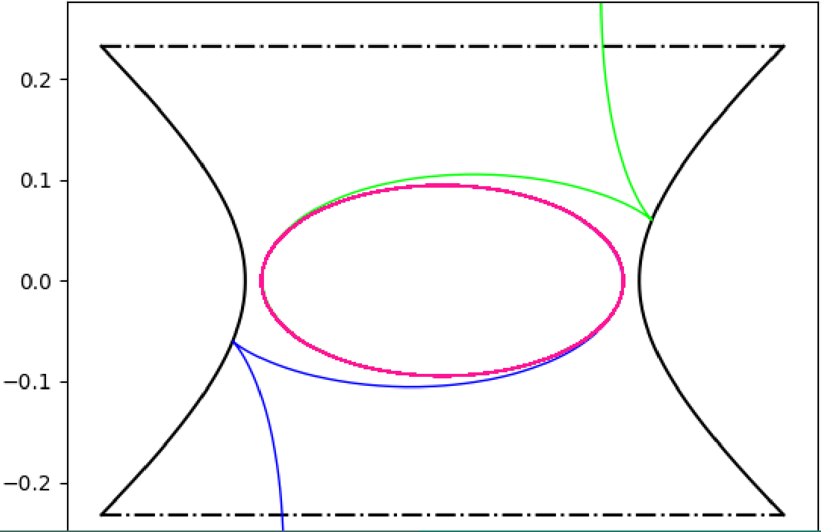
Numerical simulation - trajectory closed to localised (L1)



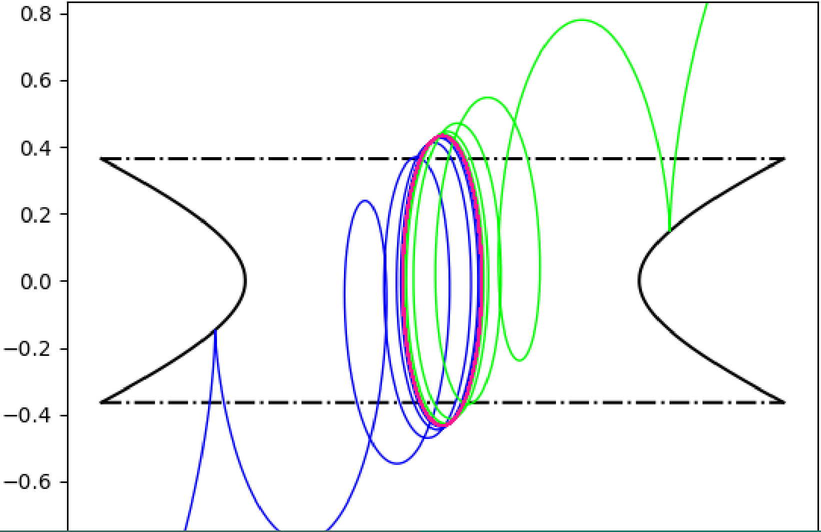
Localized solution (L1)



Localized solution (L2)



Localized solution (L3)



Three-body spatial problem

In vicinity of collinear libration points $L_i(0, x_2{}^\star, 0), i = 1, 2, 3$ equations of motion read:

$$\ddot{x}_1 = -\omega^2 x_1 + c\dot{x}_2 + f_1(\mathrm{x}, \dot{\mathrm{x}})$$

$$\ddot{x}_2 = \quad \alpha^2 x_2 - c\dot{x}_1 + f_2(\mathrm{x}, \dot{\mathrm{x}})$$

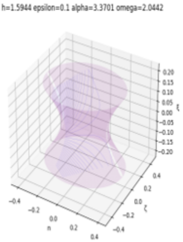
$$\ddot{x}_3 = -\frac{1}{2}(\alpha^2 - 1)x_3 + f_3(\mathrm{x}, \dot{\mathrm{x}})$$

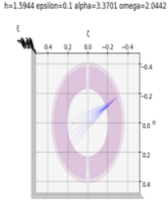
Area of possible motion for linear system:

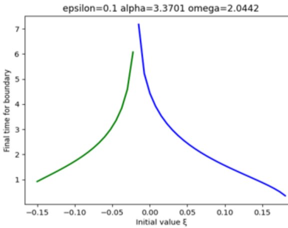
$$\omega^2 x_1^2 - \alpha^2 x_2^2 + \frac{1}{2}(\alpha^2 - 1)x_3^2 \leq 2h$$

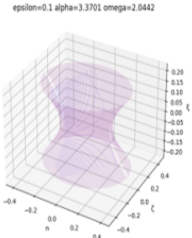
.

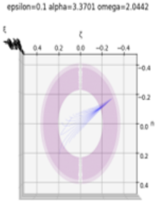
Spatial problem: L1, $\varphi = \pi/3$, autonomous / non-autonomous cases

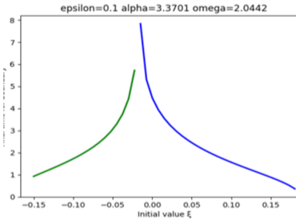




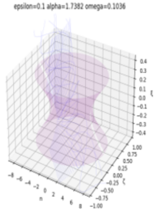


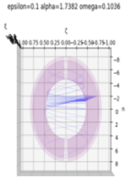


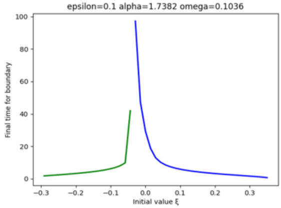


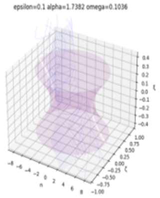


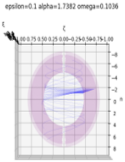
Spatial problem: L3, non-autonomous case, $\varphi = \pi/9$ and $\varphi = \pi/8$,

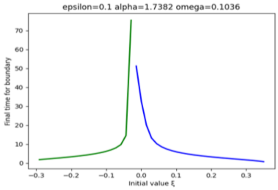












Thank you for attention!