Collocation integrator based on Legendre polynomials

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Collocation polynomial

System of ordinary differential equations:

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \tag{1}$$

$$\mathbf{y} \in \mathbb{R}^n, \qquad \mathbf{f}(t, \mathbf{y}) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n.$$

Collocation polynomial $\mathbf{u}(\tau)$ of a given degree s is used to approximate the solution:

$$\mathbf{y}(t_0 + h\tau) \approx \mathbf{u}(\tau), \qquad \mathbf{u}(\tau) : \mathbb{R} \to \mathbb{R}^n.$$

Collocation polynomial

Collocation polynomial $\mathbf{u}(\tau)$:

$$\mathbf{u}(\tau) = \sum_{k=0}^{s} \boldsymbol{\alpha}_k P_k(\tau).$$

The coefficients $\boldsymbol{\alpha}_k$ are implicitly determined by a system of nonlinear equations:

$$\sum_{k=1}^{s} \boldsymbol{\alpha}_k P_k'(c_j) = \mathbf{f}(t_0 + hc_j, \mathbf{y}_j)h, \qquad \mathbf{y}_j = \sum_{k=0}^{s} \boldsymbol{\alpha}_k P_k(c_j), \tag{2}$$

$$\boldsymbol{\alpha}_0 + \sum_{k=1}^{s} \boldsymbol{\alpha}_k P_k(0) = \mathbf{y}(t_0), \qquad c_j \in [0,1], \quad j = 1, \ldots, s.$$

Possible calculation algorithm

Let's move on to a matrix notation:

$$\begin{aligned}
\boldsymbol{\alpha}_k &\quad\rightarrow\quad \mathcal{A} \quad (n \times s).\\
\mathbf{f}(t_0 + hc_j, \mathbf{y}_j)h &\quad\rightarrow\quad \mathcal{F} \quad (n \times s).\\
P'_k(c_j) &\quad\rightarrow\quad \mathcal{C} \quad (s \times s).\\
P_k(c_j) - P_k(0) &\quad\rightarrow\quad \mathcal{B}_j \quad (s \times 1).
\end{aligned}$$

Then the system (2) can be written as

$$\mathcal{A}\mathcal{C}^T = \mathcal{F}, \qquad \mathbf{y}_j = \mathbf{y}(t_0) + \mathcal{A}\mathcal{B}_j. \tag{3}$$

or

$$\mathcal{A} = \mathcal{F}\mathcal{C}^{-T}.$$

Application conditions

The iterative process of calculating the matrix $\mathcal{A}$ will converge if

$$\left\| \frac{\partial\left(\mathcal{F}\mathcal{C}^{-T}\right)}{\partial\mathcal{A}} \right\| < 1.$$

Then we can get a restriction for $\mathbf{f}(t, \mathbf{y})$:

$$\max_p \sum_{q=1}^{n} \sum_{i=1}^{s} \left| \left(\frac{\partial f_p(t, \mathbf{y})}{\partial y^{(q)}} \right)_{\mathbf{y}=\mathbf{y}_i} \right| = \sum_{i=1}^{s} \left\| \frac{\partial \mathbf{f}(t, \mathbf{y})}{\partial \mathbf{y}} \right\|_{\mathbf{y}=\mathbf{y}_i} < \frac{1}{hsbz}, \tag{4}$$

$$b = \max_j \|\mathcal{B}_j\|, \qquad z = \max_{j,k} \left| \left(\mathcal{C}^{-T}\right)_{j,k} \right|.$$

Legendre polynomials

Shifted Legendre polynomials:

$$P_n(\tau) = \frac{1}{n!}\frac{d^n}{d\tau^n}\Big[\tau^n(\tau-1)^n\Big], \qquad P_n(0) = (-1)^n, \quad P_n(1) = 1,$$

$$-1 \le P_n(\tau) \le 1.$$

How some formulas are transformed:

$$\mathbf{y}_j = \mathbf{y}(t_0) + \mathcal{A}\mathcal{B}_j = \mathbf{y}(t_0) + \sum_{k=1}^{s} \boldsymbol{\alpha}_k \Big(P_k(c_j) - (-1)^k\Big).$$

$$\mathbf{y}(t_0+h) \approx \mathbf{u}(1) = \mathbf{y}(t_0) + \sum_{k=1}^{s} \boldsymbol{\alpha}_k \Big(1 - (-1)^k\Big) = \mathbf{y}(t_0) + \sum_{m=1}^{\lceil \frac{s}{2} \rceil} 2\boldsymbol{\alpha}_{2m-1}.$$

Lobatto quadrature nodes

Lobatto quadrature nodes are the roots of a polynomial

$$\int\limits_0^\tau P_{s-1}(\tau)d\tau = \frac{1}{(s-1)!}\frac{d^{s-2}}{d\tau^{s-2}}\Big(\tau^{s-1}(\tau-1)^{s-1}\Big).$$

Set c_j:

$$c_1 = 0, \quad c_s = 1; \qquad 0 < c_j < 1, \quad j = 2, \ldots, s-1.$$

$$\frac{1}{2} - c_j = c_{s-j} - \frac{1}{2}.$$

Application conditions

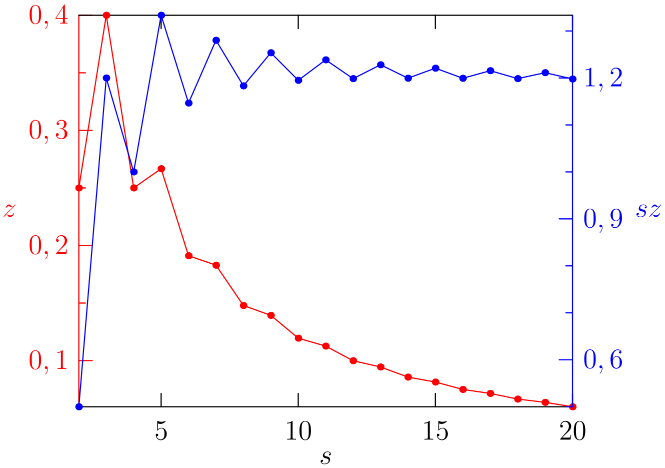
For all s we have

$$b = \max_j \|\mathcal{B}_j\| = 2.$$

$$\sum_{i=1}^{s} \left\| \frac{\partial \mathbf{f}(t,\mathbf{y})}{\partial \mathbf{y}} \right\|_{\mathbf{y}=\mathbf{y}_i} < \frac{1}{2hsz}.$$

$$\sum_{i=1}^{s} \left\| \frac{\partial \mathbf{f}(t,\mathbf{y})}{\partial \mathbf{y}} \right\|_{\mathbf{y}=\mathbf{y}_i} < \frac{1}{2.7h}.$$

Calculated dependence of z on s:



Inverse matrix study

$$\mathcal{C}^{-1}\mathcal{C} = \mathcal{E}, \qquad \text{let} \quad \mathcal{C}^{-1} \to \{z_{ki}\}_{k,i=1}^{s}, \quad \mathcal{C} \to \{P_j'(c_i)\}_{i,j=1}^{s}.$$

$$\text{then} \quad \sum_{i=1}^{s} z_{ki} P_j'(c_i) = \delta_{kj}.$$

Symmetry:

$$P_j'(1/2 - \tau) = (-1)^{j-1} P_j'(1/2 + \tau), \qquad \frac{1}{2} - c_i = c_{s-i} - \frac{1}{2}.$$

Inverse matrix study

For even $s = 2m$:

$$\text{even } j, \quad \sum_{i=1}^{m} (z_{ki} - z_{k,s-i}) P'_j(c_i) = \delta_{kj},$$

$$\text{odd } j, \quad \sum_{i=1}^{m} (z_{ki} + z_{k,s-i}) P'_j(c_i) = \delta_{kj}.$$

For odd $s = 2m + 1$:

$$\text{even } j, \quad \sum_{i=1}^{m} (z_{ki} - z_{k,s-i}) P'_j(c_i) + z_{k,m+1} P'_j(c_{m+1}) = \delta_{kj},$$

$$\text{odd } j, \quad \sum_{i=1}^{m} (z_{ki} + z_{k,s-i}) P'_j(c_i) + z_{k,m+1} P'_j(c_{m+1}) = \delta_{kj}.$$

Inverse matrix study, even $s = 2m$

For even $s = 2m$:

$$\text{even } j, \quad \sum_{i=1}^{m} (z_{ki} - z_{k,s-i}) P_j'(c_i) = \delta_{kj},$$
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For odd k:

$$z_{ki} = z_{k,s-i},$$

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For even k:

$$z_{ki} = -z_{k,s-i},$$

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For even k:

$$z_{ki} = -z_{k,s-i}, \quad z_{k,m+1} = 0,$$
$$\sum_{i=1}^{m} 2 z_{ki} P_j'(c_i) = \delta_{kj}., \quad \text{even } j.$$

Inverse matrix study

[-2.255e-17	0.1167	0.2667	0.1167	8.674e-19]
[1.406e-17	-0.1273	-5.816e-18	0.1273	2.39e-18]
[0.01389	0.04537	-0.1185	0.04537	0.01389]
[-0.025	0.03819	7.08e-18	-0.03819	0.025]
[0.01111	-0.02593	0.02963	-0.02593	0.01111]

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Thank you for your attention!

https://github.com/shvak/collo