DYNAMICS OF SELF-GRAVITATING ELLIPSOIDS

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The Dirichlet Equations

The equations of the dynamics of a homogeneous, incompressible, ideal fluid of unit density in a Lagrangian form are in the case of potential forces applied to the fluid as follows:

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right)^T \ddot{\boldsymbol{x}} = -\frac{\partial (U+p)}{\partial \boldsymbol{a}}, \tag{1}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ are the initial positions of the material points of the medium (the so-called Lagrangian coordinates), $\boldsymbol{x}(\boldsymbol{a}, t)$ are the coordinates of the points of the medium at the time t (i. e., $\boldsymbol{x}(\boldsymbol{a}, 0) = a$), $U(\boldsymbol{a}, t)$ is the density of the potential energy of the external forces, $p(\boldsymbol{a}, t)$ is the pressure, and $\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} = \left\| \frac{\partial x_i}{\partial a_j} \right\|$ is the matrix of the partial derivatives. These equations must be supplemented with the incompressibility condition, which can be written is the case at hand as

$$\det \left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\right) = 1. \tag{2}$$

Thus, we obtain a system of partial differential equations in which four quantities, viz., x_1, x_2, x_3, and p, are unknown as the functions of the variables $\boldsymbol{a}$ and t. To determine them, except initial conditions $(\boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}, \dot{\boldsymbol{x}}(\boldsymbol{a}, 0) = \boldsymbol{v}_0(\boldsymbol{a}))$, also boundary conditions must be specified; in our case, the latter reduce to the statement that the pressure has the same value independent of $\boldsymbol{a}$ everywhere on the free surface.

Dirichlet noted that, if the potential of the external forces $U(\boldsymbol{a},t)$ is a homogeneous quadratic function of the Lagrangian coordinates, i. e.

$$U(\boldsymbol{a},t) = U_0(t) + (\boldsymbol{a}, \mathbf{V}(t)\boldsymbol{a}), \tag{3}$$

where $U_0(t)$ is independent of $\boldsymbol{a}$ and $\mathbf{V}(t)$ is a symmetric matrix, then the equations of motion (1), (2) admit a partial solution

$$\boldsymbol{x}(\boldsymbol{a},t) = \mathbf{F}(t)\boldsymbol{a}, \quad \det \mathbf{F}(t) = 1. \tag{4}$$

Here, $\mathbf{F}(t)$ is a 3×3 matrix.
In this case, the boundary conditions will be satisfied provided that the fluid has initially an ellipsoidal shape,

$$(\boldsymbol{a}, \mathbf{A}_0^{-2}\boldsymbol{a}) \leqslant 1, \tag{5}$$

where $\mathbf{A}_0 = \mathrm{diag}(A_1^0, A_2^0, A_3^0)$ is the matrix of the initial semiaxes and the pressure has the form

$$p(\boldsymbol{a},t) = p_0(t) + \sigma(t)(1 - (\boldsymbol{a}, \mathbf{A}_0^{-2}\boldsymbol{a})). \tag{6}$$

We substitute (3), (4), and (6) into (1) and (2) to obtain equations for the matrix $\mathbf{F}(t)$ and the function $\sigma(t)$ in the form

$$\begin{aligned} &\mathbf{F}^T\ddot{\mathbf{F}} = -2\mathbf{V} - 2\sigma\mathbf{A}_0^{-2}, \\ &\det \mathbf{F} = 1. \end{aligned} \qquad \text{(the Dirichlet equations)} \tag{7}$$

As Dirichlet showed, the system of ten equations (7) for ten unknown functions $F_{ij}(t), \sigma(t)$, $i,j = 1,2,3$, is compatible.
Obviously, the transformation (4) changes the original ellipsoid (5) into the ellipsoid specified by the quadratic form

$$(\boldsymbol{x}, (\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T)^{-1}\boldsymbol{x}) \leqslant 1. \tag{8}$$

It is known that such a transformation is given by the orthogonal matrix

$$\zeta = \mathbf{Q}\mathbf{x}, \quad \mathbf{Q}^T = \mathbf{Q}^{-1}. \tag{9}$$

In the new coordinates ζ, the ellipsoid is specified by the relationship

$$(\zeta, \mathbf{A}^{-2}\zeta) \leqslant 1, \tag{10}$$

where $\mathbf{A} = \mathrm{diag}(A_1, A_2, A_3)$ is the matrix of the principal semiaxes at the given time.

Now, we determine the right-hand sides of equations (7). We use the known representation of the gravitational potential for the interior of the ellipsoid in the system of the principal axes

$$U(\zeta) = -\frac{3}{4} mG \int_0^\infty \frac{d\lambda}{\Delta(\lambda)} \left(1 - \sum_i \frac{\zeta_i^2}{A_i^2 + \lambda}\right), \quad \Delta^2(\lambda) = \prod_i (A_i^2 + \lambda), \tag{11}$$

where G is the gravitational constant and $m = \frac{4}{3}\pi\rho A_1 A_2 A_3$ is the mass of the ellipsoid. It is now necessary to represent (11) in terms of the elements of the transformation matrix $\mathbf{F}$ and in the Lagrangian coordinates $\boldsymbol{a}$. We use $(\mathbf{??})$ to find $\mathbf{A} = \mathbf{QFA}_0\Theta^T$ and obtain

$$\begin{aligned} &\mathbf{A}^2 = \mathbf{A}\mathbf{A}^T = \mathbf{QFA}_0^2\mathbf{F}^T\mathbf{Q}^T, \\ &\Delta^2(\lambda) = \det(\mathbf{A}^2 + \lambda\mathbf{E}) = \det(\mathbf{FA}_0^2\mathbf{F}^T + \lambda\mathbf{E}), \\ &\sum_i \frac{\zeta_i^2}{A_i^2 + \lambda} = (\zeta, (\mathbf{A}^2 + \lambda\mathbf{E})^{-1}\zeta) = (\boldsymbol{a}, \mathbf{F}^T(\mathbf{FA}_0^2\mathbf{F}^T + \lambda\mathbf{E})^{-1}\mathbf{F}\boldsymbol{a}). \end{aligned} \tag{12}$$

Thus, we find the following representation for the matrix $\mathbf{V}$ in the Dirichlet equations:

$$\mathbf{V} = \varepsilon \int_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda\mathbf{E})}} \mathbf{F}^T(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda\mathbf{E})^{-1}\mathbf{F}, \quad \varepsilon = \frac{3}{4}mG; \tag{13}$$

it can be shown by direct calculations (see [1]) that $\mathbf{V}$ depends on the elements of the matrix $\mathbf{F}$ only through symmetric combinations of the form $\Phi_{ij} = \sum_k F_{ik}F_{jk}$, which are the dot products of columns of the matrix $\mathbf{F}$.
The relationship $\frac{\partial \boldsymbol{a}}{\partial \zeta} = \mathbf{A}_0\Theta^T\mathbf{A}^{-1}$ can be used to easily show that, in the Riemann equations, $\hat{\mathbf{V}} = \mathrm{diag}(\hat{V}_1, \hat{V}_2, \hat{V}_3)$, where

$$\hat{V}_i = \varepsilon \int_0^\infty \frac{1}{\lambda + A_i^2} \frac{d\lambda}{\Delta(\lambda)} = -\frac{1}{A_i}\frac{\partial}{\partial A_i}\varepsilon \int_0^\infty \frac{d\lambda}{\Delta(\lambda)}. \tag{14}$$

[1] Dirichlet, G. L., Untersuchungen über ein Problem der Hydrodynamik (Aus dessen Nachlass hergestellt von Herrn R. Dedekind zu Zürich), *J. reine angew. Math. (Crelle's Journal)*, 1861, Bd. 58, S. 181–216.

Vorticity

We write the law of conservation of vorticity for the hydrodynamic equations in the Lagrangian form (1), thus obtaining

$$\sum_i \left(\frac{\partial x_i}{\partial a_k} \frac{\partial \dot{x}_i}{\partial a_l} - \frac{\partial x_i}{\partial a_l} \frac{\partial \dot{x}_i}{\partial a_k} \right) = \xi_{kl} = \text{const}, \tag{15}$$

with the condition $\xi_{kl} = -\xi_{lk}$ satisfied. We denote this antisymmetric matrix as $\Xi = \|\xi_{kl}\|$ and find for the Dirichlet equations (7) that

$$\Xi = \mathbf{F}^T \dot{\mathbf{F}} - \dot{\mathbf{F}}^T \mathbf{F} = \text{const}. \tag{16}$$

A straightforward proof of the conservation of vorticity Ξ based on the Dirichlet equations (7) is obvious (since the right-hand side is a symmetric matrix).
As already mentioned, the conservation of vorticity in this problem was noted by Dirichlet even before the appearance of a classical study by Helmholtz in which this law was extended to ideal hydrodynamics on the whole.

Momentum

The angular momentum relative to the center of the ellipsoid can be represented as

$$M_{ij} = \int (x_i \dot{x}_j - x_j \dot{x}_i) d^3\boldsymbol{x} = \frac{m}{5} \sum_k (F_{ik}\dot{F}_{jk} - F_{jk}\dot{F}_{ik})(A_k^0)^2. \tag{17}$$

In a matrix form, with the unimportant multiplier omitted, we have

$$\mathbf{M}' = \mathbf{F}\mathbf{A}_0^2\dot{\mathbf{F}}^T - \dot{\mathbf{F}}\mathbf{A}_0^2\mathbf{F}^T = \text{const}, \tag{18}$$

where $\mathbf{M}' = \|\frac{5}{m} M_{ij}\|$.

Energy

In addition to the linear integrals, the equations of motion also admit another, quadratic integral, viz., the total energy of the system. The integration of the kinetic and the potential energy of the fluid particles over the volume of the ellipsoid yields

$$\begin{aligned} \mathscr{E} &= \frac{m}{5}(T_e + U_e), \\ T_e &= \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{F}}\mathbf{A}_0^2\dot{\mathbf{F}}^T), \\ U_e &= -2\varepsilon \int_0^\infty \frac{d\lambda}{\sqrt{(\lambda + A_1^2)(\lambda + A_2^2)(\lambda + A_3^2)}}. \end{aligned} \tag{19}$$

The studies by L. Dirichlet in the dynamics of a self-gravitating fluid ellipsoid are dated back to 1856–1857. Ditichlet reported these studies in his lectures in 1857 and simultaneously in the *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* as a brief note [1]. Unfortunately, he had no time to describe and publish his results in full (due to his illness and untimely death in 1859). These studies were prepared for publication and posthumously published by Dedekind in 1861 [2].
Three basic results can be isolated in Dirichlet's study:

1. A new partial solution of the hydrodynamic equations is presented, which describes the motion of a homogeneous, self-gravitating ellipsoid, and the equations of motion (of fluid particles) in motionless axes are derived.
2. Seven first integrals of the obtained equations are found; six of them, linear in velocities, correspond to the conservation laws of vorticity and total momentum, and the seventh integral is the total energy of the moving fluid.
3. The motion of an axisymmetric ellipsoid is integrated in quadratures with the inclusion of Newton's and Maclaurin's spheroids as partial solutions (in this case, Dirichlet also analyzes the possibility of existence of the solutions found in the case of no external pressure, i. e., in vacuum).

[1] Dirichlet, G. L., Untersuchungen über ein Problem der Hydrodynamik, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen (Mathematisch-Physikalische Klasse)*, Jg. 1857, No. 14, Aug. 10. S. 203–207 (Dirichlet's Werke, Bd. 2, S. 28).
[2] Dirichlet, G. L., Untersuchungen über ein Problem der Hydrodynamik (Aus dessen Nachlass hergestellt von Herrn R. Dedekind zu Zürich), *J. reine angew. Math. (Crelle's Journal)*, 1861, Bd. 58, S. 181–216.

It is interesting to note that Dirichlet noted the integrals corresponding to the conservation of the vorticity vector prior to the publication of the well-known study of 1858 by Helmholtz [1]. As can be judged by the form of the obtained integrals, Dirichlet was aware (before Helmholtz) of the conservation of vorticity not only for a particular solution but also for the general hydrodynamic equations (Dirichlet's note [2] is also evidence for his awareness). This fact was also noted by Klein in his well-known lectures [3].

Dedekind, while preparing Dirichlet's results for publication, discovered the reciprocity law according to which each solution of the Ditrichlet equations is corresponded with a reciprocal solution in which the variables that describe the rotation of the ellipsoid and the fluid motion inside it are permutated; in particular, he presented a solution (the Dedekind ellipsoid) reciprocal to the Jacobi ellipsoid, with the coordinate axes remaining motionless in space and with the fluid moving inside this invariable region [4].

[1] Helmholtz, H., Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, *J. reine angew. Math.,* 1858, B. 55, S. 25-–55. Reprinted in: *Wissenschaftliche Abhandlungen von Hermann Helmholtz*, I, Barth, Leipzig, 1882, S. 101–134.
[2] Dirichlet, G. L., Untersuchungen über ein Problem der Hydrodynamik, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen (Mathematisch-Physikalische Klasse)*, Jg. 1857, No. 14, Aug. 10. S. 203–207 (Dirichlet's Werke, Bd. 2, S. 28).
[3] Klein, F., Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert (German) [Lectures on the Development of Mathematics in the 19th Century], Berlin-New York: Springer-Verlag, 1979.
[4] Dedekind, R., Zusatz zu der vorstehenden Abhandlung, *J. reine angew. Math. (Crelle's Journal)*, 1861, Bd. 58, S. 217–228.

Hamiltonian Principle and Lagrangian Formalism

It is known (see, e. g., [1]) that the motion of ideal fluid satisfies the Hamilton principle; therefore, Dirichlet's solution also satisfies this principle. This makes it possible to represent the equations of motion in a Lagrangian and, next, in a Hamiltonian form. The Hamiltonian principle for the considered problem was used for the first time by Lipschitz [2] and Padova [3].

As the Lagrangian function, it is necessary to choose the difference between the kinetic and potential energies of the fluid in the ellipsoid; within the unimportant multiplier, we have

$$L = T_e - U_e, \tag{20}$$

where T_e and U_e were defined above in (19). The elements of the matrix $\mathbf{F}$ appear as generalized coordinates.

[1] Kirchhoff, G., *Vorlesungen über mathematische Physik. Mechanik*, Leipzig: Teubner, 1876.

[2] Lipschitz, R., Reduction der Bewegung eines flüssigen homogenen Ellipsoids auf das Variationsproblem eines einfachen Integrals, und Bestimmung der Bewegung für den Grenzfall eines unendlichen elliptischen Cylinders, *J. reine angew. Math. (Crelle's Journal)*, 1874, Bd. 78, S. 245–272.

[3] Padova, E., Sul moto di un ellissoide fluido ed omogeneo, *Annali della Scuola Normale Superiore di Pisa*, t. 1, 1871, p. 1–87.

We write the Lagrange – Euler equations taking into account the constraint $\det \mathbf{F} = 1$ to obtain

$$\left(\frac{\partial L}{\partial \dot{\mathbf{F}}}\right)^{\cdot} - \frac{\partial L}{\partial \mathbf{F}} = \kappa \frac{\partial \varphi}{\partial \mathbf{F}}, \tag{21}$$

where $\varphi = \det \mathbf{F}$, and use the following matrix notation for any function: $\frac{\partial f}{\partial \mathbf{F}} = \left\| \frac{\partial f}{\partial F_{ij}} \right\|$, $\frac{\partial f}{\partial \dot{\mathbf{F}}} = \left\| \frac{\partial f}{\partial \dot{F}_{ij}} \right\|$, κ being the undefined Lagrangian multiplier. The differentiation in view of the formula $\left(\frac{\partial \varphi}{\partial \mathbf{F}}\right)^T = \varphi \mathbf{F}^{-1}$ yields

$$\ddot{\mathbf{F}}\mathbf{A}_0^2 = 2\varepsilon \frac{\partial}{\partial \mathbf{F}} \int_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{F}\mathbf{A}_0^2\mathbf{F}^T + \lambda \mathbf{E})}} + \kappa (\mathbf{F}^{-1})^T \det \mathbf{F}. \tag{22}$$

We can easily make sure that these equations coincide with the Dirichlet equations (7) if we set $\kappa = 2\sigma$.

The matrix of the initial semiaxes $\mathbf{A}_0$ appears in the Lagrangian function and the equations of motion of the system as a set of parameters. Obviously, these parameters can be transferred to the initial conditions; indeed, upon the substitution $\mathbf{G} = \mathbf{F}\mathbf{A}_0$ (suggested by Dedekind [1]), the Lagrangian function and the equation of constraint can be written as

$$L = \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{G}}\dot{\mathbf{G}}^T) + 2\varepsilon \int_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{G}\mathbf{G}^T + \lambda\mathbf{E})}}, \tag{23}$$

$$\varphi = \det\mathbf{G} = \det\mathbf{A}_0 = \mathrm{const.}$$

The initial conditions have obviously the form $\mathbf{G}|_{t=0} = \mathbf{A}_0$, and the equation of motion preserves its form, $\left(\frac{\partial L}{\partial \dot{\mathbf{G}}}\right)^{\cdot} - \frac{\partial L}{\partial \mathbf{G}} = \widetilde{\mathscr{H}}\frac{\partial \varphi}{\partial G}$.
It can also be shown that the substitution

$$\mathbf{G} \to (\det\mathbf{A}_0)^{1/3}\mathbf{G}, \quad t \to \frac{(\det\mathbf{A}_0)^{1/3}}{2\varepsilon}t$$

reduces the system (23) to the case of $\varepsilon = 1/2, \varphi = 1$. Thus, the dynamics of the self-gravitating fluid ellipsoid is described by a natural Lagrangian system without parameters on the $SL(3)$ group.

[1] Dedekind, R., Zusatz zu der vorstehenden Abhandlung, *J. reine angew. Math. (Crelle's Journal)*, 1861, Bd. 58, S. 217–228.

Symmetry Group and the Dedekind Reciprocity Law

The Lagrangian representation of the Dirichlet equations (21) offers a very simple way to finding the symmetry group of the system. Indeed, it can be shown that the Lagrangian with the constraint [see (23)] and, therefore, the equations of motion are invariant with respect to transformations of the form

$$\mathbf{G}' = \mathbf{S}_1 \mathbf{G} \mathbf{S}_2, \quad \mathbf{S}_1, \mathbf{S}_2 \in SO(3). \tag{24}$$

Thus, the system is invariant with respect to the group $\Gamma = SO(3) \otimes SO(3)$.

Clearly, the Noether integrals corresponding to the transformations (24) are the integrals of vorticity and total momentum (25). Accordingly, as will be shown below, the Riemann equations describe a system reduced based on the given symmetry group.

Furthermore, it can easily be shown using (23) that the equations of motion are invariant with respect to the discrete transformation of transposition of matrices:

$$\mathbf{G}' = \mathbf{G}^T.$$

Therefore, we have

Теорема (The Dedekind reciprocity law)

Any solution, $\mathbf{G}(t)$, of the Dirichlet equations can be placed in correspondence with the solution $\mathbf{G}'(t) = \mathbf{G}^T(t)$ for which the rotation of the ellipsoid and the rotation of the fluid inside the ellipsoid (i. e., Θ and $\mathbf{Q}$; see $(\mathbf{??})$) are interchanged.

The most widely known example is the Dedekind ellipsoid reciprocal to the Jacobi ellipsoid. In this case, the axes of the three-axial ellipsoid are spatially invariable and the fluid inside it moves around the minor axis in closed ellipses [1, 2].
The first integrals — vorticity (**??**), momentum (18), and energy (19) — can be represented in the form

$$\mathbf{\Xi} = \mathbf{G}^T\dot{\mathbf{G}} - \dot{\mathbf{G}}^T\mathbf{G}, \quad \mathbf{M} = \mathbf{G}\dot{\mathbf{G}}^T - \dot{\mathbf{G}}\mathbf{G}^T,$$

$$\mathscr{E} = \frac{1}{2}\,\mathrm{Tr}(\dot{\mathbf{G}}\dot{\mathbf{G}}^T) - 2\varepsilon\int\limits_0^\infty \frac{d\lambda}{\sqrt{\det(\mathbf{G}\mathbf{G}^T + \lambda\mathbf{E})}}. \tag{25}$$

We use a decomposition of the form

$$\mathbf{G} = \mathbf{F}\mathbf{A}_0 = \mathbf{Q}^T\mathbf{A}\Theta \tag{26}$$

We introduce the angular velocities corresponding to the orthogonal transformations,

$$\mathbf{w} = \dot{\mathbf{Q}}\mathbf{Q}^T, \quad \omega = \dot{\Theta}\Theta^T, \tag{27}$$

which are known to be antisymmetric matrices.

[1] Dedekind, R., Zusatz zu der vorstehenden Abhandlung, *J. reine angew. Math. (Crelle's Journal)*, 1861, Bd. 58, S. 217–228.

[2] Riemann, B., Ein Beitrag zu den Untersuchungen über die Bewegung einer flüssigen gleichartigen Ellipsoïdes, *Abh. d. Königl. Gesell. der Wiss. zu Göttingen*, 1861.

Riemann used the decomposition to represent the equations of motion on the configuration space $\mathbb{R}^2 \otimes \mathrm{SO}(3) \otimes \mathrm{SO}(3)$ (the direct product of the Abel group of translations and two copies of the group of rotations of three-dimensional space), with the elements of the matrices $\mathbf{w}$ and ω corresponding to the velocity components with respect to the basis of left-invariant vector fields. The equations of motion assume the form of the Poincaré equations on the Lie group [1]; in view of the fact that the Lagrangian function (23) is independent of the elements of the matrices $\mathbf{Q}$ and Θ and with due account for the constraint $\varphi = A_1 A_2 A_3 = \mathrm{const}$, we obtain the following representation of the Riemann equations:

$$\left(\frac{\partial L}{\partial \dot{A}_i}\right)^{\cdot} = \frac{\partial L}{\partial A_i} + \tilde{\kappa}\frac{\partial \varphi}{\partial A_i},$$
$$\left(\frac{\partial L}{\partial \mathrm{w}_i}\right)^{\cdot} = \sum_{j,k} \varepsilon_{ijk}\frac{\partial L}{\partial \mathrm{w}_j}\mathrm{w}_k, \quad \left(\frac{\partial L}{\partial \omega_i}\right)^{\cdot} = \sum_{j,k} \varepsilon_{ijk}\frac{\partial L}{\partial \omega_j}\omega_k. \tag{28}$$

where $\tilde{\kappa}$ is the Lagrangian undetermined multiplier (which coincides with σ within a multiplier) and ε_{ijk} is the Levi-Civita antisymmetric tensor.
From here on, the components w_i and ω_i are related to the elements of the antisymmetric matrices (27) according to the regular rule

$$\mathrm{w}_{ij} = \varepsilon_{ijk}\mathrm{w}_k, \quad \omega_{ij} = \varepsilon_{ijk}\omega_k. \tag{29}$$

[1] Borisov, A.V. and Mamaev, I.S., *Rigid Body Dynamics*, Moscow–Izhevsk: Inst. Comp. Sci., RCD, 2005 (in Russian).

An enormous contribution to the investigation of the dynamics of the fluid ellipsoid was made by an outstanding work by Riemann [1], which appeared in 1861, virtually immediately after the publication of Dirichlet's studies. The basic results of this work can be briefly formulated as follows:

- The equations of motion in moving axes (the principal axes of the ellipsoid) were obtained, so that the order of the system was lowered and a linear-integral-based reduction was done. Furthermore, Riemann represented the equations of motion of the reduced system in a Hamiltonian form with a linear Lee – Poisson bracket (Riemann himself called this procedure reduction to a better observable form).
- All partial solutions corresponding to the motion of the ellipsoid without changes in its form were presented and conditions of their existence were analyzed (i. e., the possible lengths of the major semiaxes). All these solutions imply that the ellipsoid rotates about an axis immovable in space. They included all solutions known by that time — those obtained by Newton, Maclaurin, Jacobi, and Dedekind (for which the rotational axis coincides with one of the principal axes) and also new solutions (Riemann ellipsoids) for which the rotational axis lies in one of the principal planes of the ellipsoid.

[1] Riemann, B., Ein Beitrag zu den Untersuchungen über die Bewegung einer flüssigen gleichartigen Ellipsoïdes, *Abh. d. Königl. Gesell. der Wiss. zu Göttingen*, 1861.

- Riemann used the energy integral of the system as the Lyapunov function (in modern terminology) to investigate the stability of shape-preserving motions (in the class of motions preserving the ellipsoidal shape); in this way, he found the Lyapunov-stability limits for the Maclaurin spheroids and Jacobi ellipsoids.
- A particular case was noted in which a three-axial ellipsoid (unsteadily) rotates about one of the principal axes, and its semiaxes vary with time. This gives rise to a (Hamiltonian) system with two degrees of freedom for which Riemann noted an analogy with the motion of a material point on a two-dimensional surface of the form $xyz = \mathrm{const}$ in a potential field of forces (it is this case that we will consider below in detail).

The study by Riemann was unique in terms of the importance of its results and possibilities of further generalizations; it was well in advance of its time.
There is also a study by Brioschi of 1861 [1], which was dedicated to lowering the order in the Dirichlet equations with the use of a decomposition into a potential and a vortical component. However, no substantial advance in the problem was associated with this work.

[1] Brioschi, F., Développements relatifs au § 3 des Recherches de *Dirichlet* sur un problème d'Hydrodynamique, vol. 58, pag. 181 et suivantes de ce Journal, *J. reine angew. Math. (Crelle's Journal)*, 1861, Bd. 59, S. 63–73.

In his lectures in mechanics of 1876, Kirchhoff [1] also considered the motion of self-gravitating fluid ellipsoids. He noted that the d'Alembert principle is applicable to the Dirichlet motion (although he did not use it to derive the equation of motion). Kirchhoff presents a quadrature for the axisymmetric case and separately analyzes the case where the ellipsoid preserves the directions of its axes in space (a particular case of the motion considered by Riemann); Kirchhoff (following Riemann) conjectures that this problem also cannot be integrated in quadratures.

The possibility of applying the variational principle to the derivation of the equations of motion of a fluid ellipsoid was independently shown by Padova in 1871 [2] and Lipschitz in 1874 [3]. In the latter study [3], the problem of the motion of an elliptic cylinder was also formulated and integrated in quadratures.

[1] Kirchhoff, G., *Vorlesungen über mathematische Physik. Mechanik*, Leipzig: Teubner, 1876.

[2] Padova, E., Sul moto di un ellissoide fluido ed omogeneo, *Annali della Scuola Normale Superiore di Pisa*, t. 1, 1871, p. 1–87.

[3] Lipschitz, R., Reduction der Bewegung eines flüssigen homogenen Ellipsoids auf das Variationsproblem eines einfachen Integrals, und Bestimmung der Bewegung für den Grenzfall eines unendlichen elliptischen Cylinders, *J. reine angew. Math. (Crelle's Journal)*, 1874, Bd. 78, S. 245–272.

Betti [1] also used the variational principle to derive the equations of motion of a fluid ellipsoid and represented these equations in a Lagrangian and a Hamiltonian form. However, as Tedone noted in his extensive survey [2], Betti made a mistake in his study when applying the variational principle to the derivation of the equation of motion of a homogeneous ellipsoid with an ellipsoidal fluid-density stratification. In this case, the hydrodynamic equations for the stratified, self-gravitating ellipsoid do not admit a solution with a linear dependence on the initial coordinates, which Betti considered (in view of the complex dependence of the gravitational potential inside the stratified ellipsoid). Nevertheless, all Betti's results remain valid for a constant density. Betti also represented the equations of motion in a Hamiltonian form (explicitly using the Poisson brackets on the $so(3)$ algebra) with a linear Poisson bracket and carried out a linear-integral-based reduction.

[1] Betti, E., Sopra i moti che conservano la figura ellissoidale a una massa fluida eterogenea, *Annali di Matematica Pura ed Applicata, Serie II*, 1881, vol. X, pp. 173–187.

[2] Tedone, O., Il moto di un ellissoide fluido secondo l'ipotesi di Dirichlet, *Annali della Scuola Normale Superiore di Pisa*, 1895, t. 7, pp. I–IV+1–100.

The above-listed results are the principal achievements of the classical period of the investigation of the dynamics of the Dirichlet ellipsoids.

General problems of the dynamics and statics of fluid ellipsoids, including the issues of stability, were investigated in classical treatises by Basset [1], Lamb [2], Thomson and Tait [3], Routh [4], in books by Appell [5], Lyttleton [6], in certain studies by Basset [7–9], Duhem [10], Hagen [11], Hicks [12], Hill [13], Love [14, 15], etc. Note also the following related subjects that constitute particular lines of research in this area.

[1] Basset, A., *A Treatise on Hydrodynamics: With Numerous Examples*, Vol. II, Ch. 15., Cambridge: Deighton, Bell and Co., 1888.
[2] Lamb, H., *Hydrodynamics*, New York: Dover Publications, 1932.
[3] Thomson, W. and Tait, P. G., *Treatise on Natural Philosophy*, Cambridge University Press, Part II, 1912 (first edition 1883).
[4] Routh, E. J., *A Treatise on Analytical Statics*, Cambridge: Cambridge University Press, 1922, Vol. 2.
[5] Appell, P., *Traité de Mécanique rationnelle*, tome IV: *Figures d'équilibre d'une masse liquide homogène en rotation*, 2 ed., Paris: Gauthier-Villars, 1932 (IV-1), 1937 (IV-2).
[6] Lyttleton, R. A., *The Stability of Rotating Liquid Masses*, Cambridge: Cambridge Univ. Press, 1953.
[7] Basset, A., On the Motion of a Liquid Ellipsoid under the Influence of its Own Attraction, *Proc. London Math. Soc.*, 1885, vol. s1-17, no. 1, pp. 255–262.
[8] Basset, A., On the Stability of a Liquid Ellipsoid which is Rotating about a Principal Axis under the Influence of its Own Attraction, *Proc. London Math. Soc.*, 1887, s1-19, pp. 46–56.
[9] Basset, A., On the Steady Motion of an Annular Mass of Rotating Liquid, *Amer. J. Math.*, 1889, vol. 11, no. 2, pp. 172–181.
[10] Duhem, M. P., Sur la stabilite de l'équilibre relatif d'une masse fluide animée d'un mouvement de rotation, *J. de Math. Pures et Appl.*, 1905, vol. 7, ser. 5, pp. 331–350.

[11] Hagen, J., Ueber die Stabilitat des Gleichgewichtes einer auf einem dreiaxigen Ellipsoid mit kleinen Excentricitaten ausgebreiteten Flussigkeit, *Zeitschrift für Mathematik und Physik*, 1877, vol. 22, pp. 65–86.
[12] Hicks, W. M., On the Motion of a Mass of Liquid under its Own Attraction, when the Initial Form is an Ellipsoid, *Proc. Camb. Phil. Soc.*, 1883, Vol. IV, Pt. VI, pp. 1–4.
[13] Hill, M. J. M., Note on the Motion of a Fluid Ellipsoid under its Own Attraction, *Proc. London Math. Soc.*, 1891, s1-23, pp. 88–95.
[14] Love, A. E. H., On the Motion of a Liquid Elliptic Cylinder under its Own Attraction, *Quart. J. of Pure and Appl. Math.*, 1889, vol. 23, pp. 153–158.
[15] Love, A.E.H., The Oscillations of a Mass of Gravitating Liquid in the Form of an Elliptic Cylinder which Rotates as if Rigid about its Axis,*Quart. J. of Pure and Appl. Math.*, n.d., pp. 158–165.

We represent the Riemann equations in a Hamiltonian form. To this end, we first use the constraint equation $\varphi = \mathrm{const}$ to find a representation of one semiaxis,

$$A_3 = \frac{v_0}{A_1 A_2}, \tag{30}$$

where v_0 is the volume of the ellipsoid (within a multiplier). We carry out the Legendre transformation

$$\begin{gathered} p_i = \frac{\partial L}{\partial \dot{A}_i}, \quad m_k = \frac{\partial L}{\partial \mathrm{w}_k}, \quad \mu_k = \frac{\partial L}{\partial \omega_k}, \quad i = 1,2, \quad k = 1,2,3, \\ H = \sum_i p_i \dot{A}_i + \sum_k (m_k \mathrm{w}_k + \mu_k \omega_k) - L\,|_{\dot{A},\omega,\mathrm{w} \to p,m,\mu} . \end{gathered} \tag{31}$$

It can be shown using the expressions for the integrals, that the vectors $\boldsymbol{m} = (m_1, m_2, m_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ are related to the momentum and vorticity of the ellipsoid via the formulas

$$\boldsymbol{m} = \mathbf{Q}^T \boldsymbol{M}', \qquad \mu = \Theta^T \xi', \tag{32}$$

where the vectors $\boldsymbol{M}'$ and ξ' are constituted by the components of the antisymmetric matrices $\mathbf{M}'$ and Ξ' according to the normal rule (29). In the new variables, the equations of motion assume the form

$$\dot{A}_i = \frac{\partial H}{\partial \dot{p}_i}, \quad \dot{p}_i = \frac{\partial H}{\partial A_i}, \quad i = 1,2,$$
$$\dot{\boldsymbol{m}} = \boldsymbol{m} \times \frac{\partial H}{\partial \boldsymbol{m}}, \quad \dot{\mu} = \mu \times \frac{\partial H}{\partial \mu}. \tag{33}$$

Here, the Hamiltonian is

$$H = H_A + H_{m\mu} + U_e,$$
$$H_A = \frac{1}{2} \frac{A_3^{-2}(p_1^2 + p_2^2) + (p_1 A_2^{-1} - p_2 A_1^{-1})^2}{\sum A_i^{-2}}, \tag{34}$$
$$H_{m\mu} = \frac{1}{4} \sum_{\text{cycle}} \left(\frac{m_i + \mu_i}{A_j - A_k} \right)^2 + \left(\frac{m_i - \mu_i}{A_j + A_k} \right)^2,$$

where U_e is specified by formula (19) and it is assumed that A_3 is defined according to (30).

In addition, equations (33) must necessarily be supplemented with equations describing the evolution of the matrices $\mathbf{Q}$ and Θ; they have the form

$$\dot{Q}_{ij} = \sum_{k,l} \varepsilon_{ikl} Q_{kj} \frac{\partial H}{\partial m_l}, \quad \dot{\Theta}_{ij} = \sum_{k,l} \varepsilon_{ikl} \Theta_{kj} \frac{\partial H}{\partial \mu_l}. \tag{35}$$

Equations (33) and (35) form a Hamiltonian system with eight degrees of freedom and uncanonical Poisson brackets,

$$\{A_i, p_j\} = \delta_{ij}, \quad \{m_i, m_j\} = \varepsilon_{ijk} m_k, \quad \{\mu_i, \mu_j\} = \varepsilon_{ijk} \mu_k, \tag{36}$$

$$\{m_r, Q_{jk}\} = \varepsilon_{ikl} Q_{jl}, \quad \{\mu_i, \Theta_{jk}\} = \varepsilon_{ikl} \Theta_{jl}, \tag{37}$$

where zero brackets are omitted.

Remark

The elimination of one semiaxis (30) *results in the loss of symmetry of the Hamiltonian* (34)*; therefore, the equations for the semiaxes* A_i *are normally left in the Lagrangian form with an undetermined multiplier [1, 2].*

[1] Riemann, B., Ein Beitrag zu den Untersuchungen über die Bewegung einer flüssigen gleichartigen Ellipsoïdes, *Abh. d. Königl. Gesell. der Wiss. zu Göttingen*, 1861.

[2] Chandrasekhar S., *Ellipsoidal Figures of Equilibrium*, New Haven, London: Yale University Press, 1969.

It can be seen from the above relationships that the system of equations (33), which describes the evolution of the variables $A_i, p_i, \boldsymbol{m}$, and μ, separates; in addition, the Poisson bracket of these variables, (36), also proves to be closed. It is not difficult to show that that equations (33) describe a system reduced over the symmetry group (24). Limitation: the brackets (36) obviously have two Casimir functions,

$$\Phi_m = (\boldsymbol{m}, \boldsymbol{m}), \quad \Phi_\mu = (\mu, \mu), \tag{38}$$

and have a rank of eight (provided that $\Phi_m \neq 0, \Phi_\mu \neq 0$).
Therefore, the reduced system has generally four degrees of freedom.
In particular cases where *one of the integrals* (38) *is zero, the reduced system has three degrees of freedom*. These are so-called irrotational ($\Phi_\mu = 0$) and momentum-free ($\Phi_m = 0$) ellipsoids.
If both of the integrals (38) *vanish, the reduced system has two degrees of freedom* and describes oscillations of the ellipsoid without changes in the directions of the axes and without inner flows (this case will be considered below in detail).

Axisymmetric Case (Dirichlet [1])

It can easily be shown that the equations of motion determined by the Lagrangian function (23) admit a (two-dimensional) invariant manifold that consists of matrices of the form

$$\mathbf{G} = \begin{Vmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & w \end{Vmatrix},$$

where $\det \mathbf{G} = (u^2 + v^2)w = v_0 = \mathrm{const}$ is the volume of the ellipsoid. This manifold corresponds to an axisymmetric motion of the fluid ellipsoid (see [1]). In this case, the matrix of the principal semiaxes is

$$\mathbf{A} = (\mathbf{G}\mathbf{G}^T)^{1/2} = \mathrm{diag}(\sqrt{u^2 + v^2}, \sqrt{u^2 + v^2}, w).$$

In view of the condition $\det \mathbf{G} = v_0$, we make the substitution of variables

$$u = v_0^{1/3} r \cos \psi, \quad v = v_0^{1/3} r \sin \psi, \quad w = \frac{v_0^{1/3}}{r^2}$$

and find that the Lagrangian function (23) is

$$L = v_0^{2/3} \left(\left(1 + \frac{2}{r^6} \right) \dot{r}^2 + r^2 \dot{\psi}^2 + U_s \right),$$

[1] Dirichlet, G. L., Untersuchungen über ein Problem der Hydrodynamik (Aus dessen Nachlass hergestellt von Herrn R. Dedekind zu Zürich), *J. reine angew. Math. (Crelle's Journal)*, 1861, Bd. 58, S. 181–216.

where

$$U_s = -\frac{2\varepsilon}{v_0}\int\limits_0^\infty \frac{d\lambda}{(\lambda + r^2)\sqrt{\lambda + 1/r^4}} = -\frac{2\varepsilon}{v_0} r^2 \times \begin{cases} \dfrac{2\operatorname{arctg}\sqrt{r^6 - 1}}{\sqrt{r^6 - 1}}, & r > 1, \\ \dfrac{\ln\left(\frac{1+\sqrt{1-r^6}}{1-\sqrt{1-r^6}}\right)}{\sqrt{1-r^6}}, & r < 1. \end{cases}$$

The variable ψ is cyclic; therefore, we have a first integral of the form

$$p_\psi = \frac{1}{v_0^{2/3}}\frac{\partial L}{\partial \dot\psi} = 2r^2\dot\psi,$$

which coincides within a multiplier with the single nonzero component of the momentum M'_{12} (18). With the use of the energy integral (19), we obtain a quadrature that specifies the evolution of r:

$$\left(1 + \frac{2}{r^6}\right)\dot r^2 = h - U_*, \quad U_* = U_s + \frac{c}{r^2},$$

where $h = \frac{\mathscr{E}}{m v_0^{2/3}}$ and $c = \frac{p_\psi}{4}$ are fixed values of the energy and momentum integrals. The minimum of the reduced potential U_* corresponds to the Maclaurin spheroid.

Riemannian Case [1]

There is an invariant manifold more general than the above-described one. It is specified by the block–diagonal matrix of the general form

$$\mathbf{G} = \begin{Vmatrix} u_1 & v_1 & 0 \\ u_2 & v_2 & 0 \\ 0 & 0 & w_3 \end{Vmatrix}. \qquad (39)$$

We compute the integrals (**??**) and (18) obtaining

$$M'_{12} = u_1\dot{u}_2 - u_2\dot{u}_1 + v_1\dot{v}_2 - v_2\dot{v}_1, \quad M'_{23} = M'_{13} = 0,$$
$$\xi'_{12} = u_1\dot{v}_1 - v_1\dot{u}_1 + u_2\dot{v}_2 - v_2\dot{u}_2, \quad \xi'_{23} = \xi'_{13} = 0,$$

It is also obvious that $\mathrm{i}\mathbf{Q}$ and Θ have in this case a block–diagonal form similar to (39); therefore, this case corresponds to that noted by Riemann, for which, in equations (33), we should set

$$m_1 = m_2 = 0, \qquad m_3 = \text{const},$$
$$\mu_1 = \mu_2 = 0, \qquad \mu_3 = \text{const}.$$

[1] Riemann, B., Ein Beitrag zu den Untersuchungen über die Bewegung einer flüssigen gleichartigen Ellipsoïdes, *Abh. d. Königl. Gesell. der Wiss. zu Göttingen*, 1861.

Thus, we obtain a Hamiltonian system with two degrees of freedom, which describes the evolution of the principal semiaxes A_1 and A_2; its Hamiltonian is

$$H = \frac{1}{2}\frac{A_3^{-2}(p_1^2+p_2^2)+(p_1A_2^{-1}-p_2A_1^{-1})^2}{\sum A_i^{-2}} + U_*(A_1,A_2), \qquad (40)$$

where the reduced potential is

$$U_* = U_e + \frac{c_1^2}{(A_1-A_2)^2} + \frac{c_2^2}{(A_1+A_2)^2},$$

and $c_1^2 = \frac{1}{4}(m_3+\mu_3)^2$, $c_2^2 = \frac{1}{4}(m_3-\mu_3)^2$ are fixed constants of the integrals.
The particular version of the system (40) for $c_1 = c_2 = 0$ (i. e., for invariable directions of the principal axes of the ellipsoid) was also noted by Kirchhoff [1], who suggested that the problem does not reduce to quadratures.
At $U_* = 0$, the Hamiltonian (40) describes a geodesic flow on the cubic $A_1A_2A_3 = \mathrm{const.}$
This remarkable analogy between two different dynamical systems was also noted by Riemann.

[1] Kirchhoff, G., *Vorlesungen über mathematische Physik. Mechanik*, Leipzig: Teubner, 1876.

Elliptic Cylinder (Lipschitz [1])

This case can be obtained through a limiting process in the Riemannian case, with one axis of the ellipsoid going to infinity ($A_3 \to \infty$). It is, however, more convenient to start with considering the case of a two-dimensional motion of fluid assuming that the matrix $\mathbf{F}$ has the form

$$\mathbf{F} = \left\| \begin{array}{c|c} \bar{\mathbf{F}} & 0 \\ \hline 0 & 1 \end{array} \right\|, \qquad \det \bar{\mathbf{F}} = 1, \tag{41}$$

where $\bar{\mathbf{F}}$ is a 2×2 matrix with unit determinant.

Obviously, the considerations on which the derivation of the Dirichlet equations [2] was based can be applied to this case without modifications; only the right-hand side of the equations should be properly changed. To this end, it is necessary to use the well-known representation of the potential of the interior points of the elliptic cylinder with a large length l in the system of principal axes

$$U(\zeta) = \bar{\varepsilon} \left(U_0(l) - \frac{\zeta_1^2}{A_1(A_1+A_2)} - \frac{\zeta_2^2}{A_2(A_1+A_2)} \right) + O(1/l),$$

where $\bar{\varepsilon} = G\bar{m}$, G is the gravitational constant and $\bar{m} = \pi \rho A_1 A_2$ is the mass per unit length of the cylinder. The constant $U_0(l) \underset{l\to\infty}{\longrightarrow} \infty$ does not appear in the equations of motion and can be omitted.

[1] Lipschitz, R., Reduction der Bewegung eines flüssigen homogenen Ellipsoids auf das Variationsproblem eines einfachen Integrals, und Bestimmung der Bewegung für den Grenzfall eines unendlichen elliptischen Cylinders, *J. reine angew. Math. (Crelle's Journal)*, 1874, Bd. 78, S. 245–272.

[2] Lyapunov, A.M., Collected Works, *Collected Works, Vol. 5,*, Moscow: Izd. Akad. Nauk, 1965

By analogy with the above considerations, we pass to the Lagrangian representation and make the substitution $\bar{\mathbf{G}} = \bar{\mathbf{F}}\bar{\mathbf{A}}_0$, where $\bar{\mathbf{A}}_0 = \mathrm{diag}(A_1^0, A_2^0)$, to obtain the Lagrangian of the system in the form

$$L = \frac{1}{2}\,\mathrm{Tr}\left(\dot{\bar{\mathbf{G}}}\dot{\bar{\mathbf{G}}}^T\right) - \bar{U}_e,$$
$$\bar{U}_e = -2\bar{\varepsilon}\ln(A_1 + A_2)^2 = -2\bar{\varepsilon}\ln(\mathrm{Tr}(\bar{\mathbf{G}}\bar{\mathbf{G}}^T) + 2\det\bar{\mathbf{G}}).$$

Based on the singular decomposition of the matrix $\bar{\mathbf{G}} = \bar{\mathbf{Q}}^T\bar{\mathbf{A}}\bar{\Theta}$ with

$$\bar{\mathbf{Q}} = \begin{Vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{Vmatrix}, \quad \bar{\Theta} = \begin{Vmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{Vmatrix}, \quad \mathbf{A} = \begin{Vmatrix} A_1 & 0 \\ 0 & A_2 \end{Vmatrix},$$

explicitly substituted, we obtain a Lagrangian function in the form

$$L = \frac{1}{2}\left(\dot{A}_1^2 + \dot{A}_2^2 + (A_1\dot{\phi} - A_2\dot{\psi})^2 + (A_2\dot{\phi} - A_1\dot{\psi})^2\right) - \bar{U}_e(A_1, A_2).$$

We can see that the variables ϕ and ψ are cyclic, and there are two linear integrals

$$\frac{\partial L}{\partial\dot{\phi}} = p_\phi, \quad \frac{\partial L}{\partial\dot{\psi}} = p_\psi. \tag{42}$$

We parametrize the relationship $A_1 A_2 = \bar{v}_0$ using hyperbolic functions,

$$A_1 = \bar{v}_0^{1/2}(\operatorname{ch} u + \operatorname{sh} u), \quad A_2 = \bar{v}_0^{1/2}(\operatorname{ch} u - \operatorname{sh} u).$$

We use the energy integral and the integrals (42) to obtain a quadrature for the variable u:

$$\bar{v}_0(\operatorname{ch} 2u)\dot{u}^2 = h - \bar{U}_*,$$

$$\bar{U}_* = 2\bar{\varepsilon}\ln(\operatorname{ch} u) + \frac{\bar{c}_1^2}{\operatorname{ch}^2 u} + \frac{\bar{c}_2^2}{\operatorname{sh}^2 u},$$

where $\bar{c}_1^2 = \frac{1}{16}(p_\phi - p_\psi)^2$, $\bar{c}_2^2 = \frac{1}{16}(p_\phi + p_\psi)^2$, and h are fixed constants of the first integrals.

Let us consider in more detail the oscillations (pulsations) of a fluid ellipsoid in the Riemannian case (39). We now represent the equations of motion of the system (40) in a Hamiltonian form most convenient for a numerical investigation of the system. We parametrize the surface $A_1A_2A_3 = v_0$ using cylindrical coordinates

$$\begin{gathered} A_1 = r\cos\phi, \quad A_2 = r\sin\phi, \quad A_3 = \frac{2v_0}{r^2\sin^2 2\phi}, \\ p_1 = p_r\cos\phi - \frac{p_\phi}{r}\sin\phi, \quad p_2 = p_r\sin\phi - \frac{p_\phi}{r}\cos\phi \end{gathered}. \tag{43}$$

The Hamiltonian (40) can be represented in the form

$$H = \frac{1}{2}\left(1 + \frac{c_0^2}{r^6\sin^4 2\phi}\right)^{-1}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \frac{c_0^2}{r^6\sin^4 2\phi}\left(p_r\cos 2\phi - \frac{p_\phi}{r}\sin 2\phi\right)^2\right) + U_*(r,\phi), \tag{44}$$

where $c_0 = 4v_0$.
Since the original system is defined in the quadrant $A_1 > 0, A_2 > 0, A_3 > 0$, for this case we have $0 < \phi < \pi/2$. In this system, the transformation of variables

$$\rho = r^2, \quad \psi = 2\phi, \tag{45}$$

enables obtaining the Hamiltonian in the form

$$H=\frac{2(\rho^2(c_0^2\cos^2\psi+\rho^3\sin^4\psi)p_\rho^2+\sin^2\psi(c_0^2+\rho^3\sin^2\psi)p_\phi^2-2\rho c_0^2\cos\psi\sin\psi p_\psi p_\phi)}{\rho(c_0^2+\rho^3\sin^4\psi)}+$$
$$+U_*(\rho,\psi). \tag{46}$$

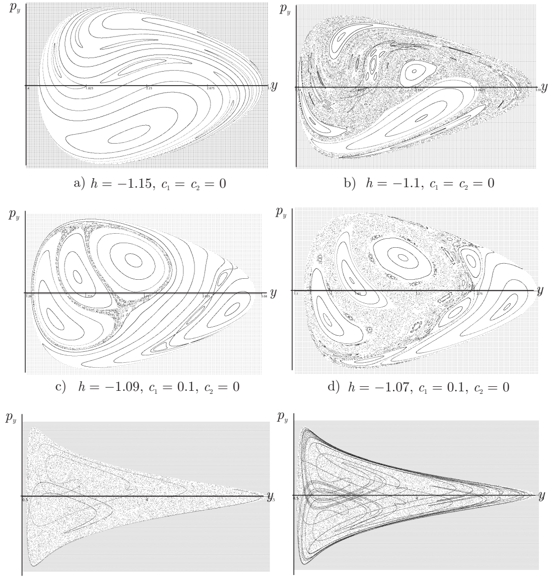
Upon passing to new Cartesian coordinates according to the formulas

$$x=\rho\cos\psi,\quad y=\rho\sin\psi, \tag{47}$$

we obtain

$$H=2\rho\left(p_x^2+\frac{y^4p_y^2}{y^4+\bar{c}_0^2\rho}\right)+U_*(x,y), \tag{48}$$

where $\rho=\sqrt{x^2+y^2}$; obviously, the system (48) is defined in the upper semiplane $(y>0)$. In this case, as we can see, the kinetic energy of the system has the simplest form.



a) $h = -1.15,\ c_1 = c_2 = 0$ b) $h = -1.1,\ c_1 = c_2 = 0$

c) $h = -1.09,\ c_1 = 0.1,\ c_2 = 0$ d) $h = -1.07,\ c_1 = 0.1,\ c_2 = 0$

e) $h = -0.9,\ c_1 = c_2 = 0$ f) $h = -0.9,\ c_1 = c_2 = 0$

Рис. 1: The Poincaré map of system (48). For all panels, $c_0 = 1, \varepsilon = 0{,}6$; for the map, the planes $x = 1$ (a–d) and $x = 0.1$ (e ,f) are chosen.

Dynamics of a Gas Cloud with Ellipsoidal Stratification

The investigation of the dynamics of gas ellipsoids traces back to a study by L. V. Ovsyannikov [1] (1956), who analyzed the most general equations describing the motion of an ideal polytropic gas, without taking into account gravitation, with a velocity field linear in the coordinates of the gas particles (from here on, by the gas ellipsoid, we mean the Dirichlet solution generalized to various models of compressible fluid). Note that the paper [1] is very brief and purely mathematical: in fact, the equations of motion are obtained there, several possible cases of the existence of the considered solution are noted, and an incomplete set of first integrals is given for two cases. It is interesting that Ovsyannikov's paper contains no references, so that the relationship between the obtained solution and the Dirichlet solution is not revealed.

Later, D. Lynden-Bell [2] (1962) demonstrated, also without any references, the existence of the solution in the form of a spheroid for a self-gravitating dust cloud (i. e., for a medium not resisting to deformations, $p \equiv 0$).

[1] Ovsyannikov, L.V., A New Solution of the Equations of Hydrodynamics, *Dokl. Akad. Nauk SSSR (N.S.)*, 1956, vol. 111, pp. 47–49 (in Russian).
[2] Lynden-Bell, D., On the Gravitational Collapse of a Cold Rotating Gas Cloud, *Proc. Camb. Phys. Soc.*, 1962, vol. 58, pp. 709–711.

Ya. B. Zel'dovich [1] (1965) obtained the equations of motion of a self-gravitating dust ellipsoid in the general case and studied (on a physical level of rigor) the possibility of collapse and expansion in this problem. Likely, Ya. B. Zel'dovich also overlooked the relationship of this problem to the Dirichlet – Riemann problem, since the model of a dust cloud can be obtained simply by setting $p = 0$ in the Dirichlet equations.
Independently of Ovsyannikov (at least without a reference), F. Dyson [2] (1968) obtained the equations of motion of an ideal-gas cloud in the case of an isothermal flow (although without the assumption of a polytropic behavior of the gas); a Gaussian density distribution with an ellipsoidal stratification was found. Dyson noted a relationship between the obtained solution and the Dirichlet problem and wrote the equations of motion of the gas ellipsoid in a Riemannian form.

[1] Zel'dovich, Ya. B., Newtonian and Einsteinian Motion of Homogeneous Matter, *Astronom. Zh.*, 1964, vol. 41, no. 5, pp. 872–883 [*Soviet Astronomy*, 1964, vol. 8, no. 5].

[2] Dyson, F. J., Dynamics of a Spinning Gas Cloud, *J. Math. Mech.*, 1968, vol. 18, no. 1, pp. 91–101.

Also independently of Ovsyannikov, M. Fujimoto [1] (1968) describes a model of a cooling ellipsoidal gas cloud; in essence, he obtains a generalization of a case considered by Ovsyannikov (if we assume the cooling parameter to be $æ = 0$, we will obtain Ovsyannikov's equations). In addition, in Fujimoto's model, the density is constant, which enabled taking into account the gravitational interaction between the particles of the cloud. Fujimoto also noted a relationship of this problem to the Dirichlet problem and, in studying it, used the techniques developed by Chandrasekhar [2] and Rossner [3].

[1] Fujimoto, F., Gravitational Collapse of Rotating Gaseous Ellipsoids, *Astrophys. J.*, 1968, vol. 152, no. 2 pp. 523–536.

[2] Chandrasekhar S., *Ellipsoidal Figures of Equilibrium*, New Haven, London: Yale University Press, 1969.

[3] Rossner, L. F., The Finite-amplitude Oscillations of the Maclaurin Spheroids, *Astrophys. J.*, 1967, vol. 149, pp. 145–168.

Let us also mention a study by Anisimov [1] (1970), who follows [2] and [3] considering two cases of the integrable dynamics of a gas ellipsoid without allowances for gravitation but with the additional condition of the monoatomic structure of the gas (a polytropic index of $\gamma = \frac{5}{3}$). The first case is the motion of an axisymmetric ellipsoid; the second, of an elliptic cylinder. A nonautonomic Jacobi integral was found (which is due to the uniformity of the potential with a uniformity degree of -2). This integral is essentially necessary for integration in the cases under study; as we will show below, these systems are not integrable in the general case.

[1] Anisimov, S.I. and Lysikov, Iu.I, Expansion of a Gas Cloud in Vacuum, *Prikl. mat. mekh.,* 1970, vol. 34, no. 5, pp. 926–929 [*J. Appl. Math. Mech.,* 1970, vol. 34, no. 5, pp. 882–885].
[2] Ovsyannikov, L.V., A New Solution of the Equations of Hydrodynamics, *Dokl. Akad. Nauk SSSR (N.S.)*, 1956, vol. 111, pp. 47–49 (in Russian).
[3] Dyson, F. J., Dynamics of a Spinning Gas Cloud, *J. Math. Mech.,* 1968, vol. 18, no. 1, pp. 91–101.

Bogoyavlenskiĭ [1] (1976) analyzes the dynamics of a gas ellipsoid on a physical level of rigor taking into account gravitation (i. e., he considers the Fujimoto model without cooling). Explicit Lagrangian and Hamiltonian representations of the system are used. Gaffet [2–4] shows that the system that describes irrotational gas ellipsoids without considering gravitation, for a monoatomic gas $\left(\gamma = \frac{5}{3}\right)$, satisfies the Painlevé property; in these studies, first integrals are found and integration in quadratures is carried out for certain particular cases.

[1] Bogoyavlenskij, O.I., Dynamics of a gravitating gaseous ellipsoid, *Prikl. mat. mekh.*, 1976, vol. 40, no. 2, pp. 270–280 [*J. Appl. Math. Mech.,* 1976, vol. 40, no. 2, pp. 246–256].
[2] Gaffet, B., Expanding Gas Clouds of Ellipsoidal Shape: New Exact Solutions, *J. Fluid Mech.*, 1996, vol. 325, pp. 113–144.
[3] Gaffet, B., Sprinning Gas without Vorticity: the Two Missing Integrals, *J. Phys. A: Math. Gen.*, 2001, vol. 34, pp. 2087–2095.
[4] Gaffet, B., Sprinning Gas Clouds: Liouville Integrability, *J. Phys. A: Math. Gen.*, 2001, vol. 34, pp. 2097–2109.

There are also studies analyzing a spherically symmetric motion of a gas cloud; one of the most general solutions is described by Lidov [1], who considers time-dependent, one-dimensional, spherically symmetric, adiabatic motions of a self-gravitating mass of a perfect gas.
Nemchinov [2] uses a solution that describes the ellipsoidal expansion of a gas cloud to find characteristic features of nonspherical explosions (in particular, he notes an increase in the impact of the stream in the direction of one of the principal axes compared to a similar spherical explosion); the effect of the heating of the cloud on the expansion speed is also investigated.
Finally, let us mention a series of studies (see [3] and references therein) generalizing the problem of the expansion of an ellipsoidal cloud to vacuum (or the collapse of an ellipsoidal cavity) with the presence of a rarefaction (compression) wave.

[1] Lidov, M.L., Exact Solutions of the Equations of One-dimensional Unsteady Motion of a Gas, Taking Account of the Forces of Newtonian Attraction, *Doklady Akad. Nauk SSSR (N.S.)*, 1951, vol. 97, pp. 409–410 (in Russian).
[2] Nemchinov, I.V., Expansion of a Tri-axial Gas Ellipsoid in a Regular Behavior, *Prikl. mat. mekh.*, 1965, vol. 29, no. 1, pp. 134—140 [*J. Appl. Math. Mech.,* 1965, vol. 29, no. 1, pp. 143–150].
[3] Deryabin, S.L., One-Dimension Escape of Self-Gravitating Ideal Gas Into Vacuum, *Computational technolgies*, 2003, vol. 8, no. 4, pp. 32–44.

FIGURES OF EQUILIBRIUM OF AN INHOMOGENEOUS SELF-GRAVITATING FLUID

Bizyaev, I. A., Borisov, A. V., and Mamaev, I. S., Figures of Equilibrium of an Inhomogeneous Self-Gravitating Fluid, *Celest. Mech. Dyn. Astr.* 2015, vol 122, pp. 1–26.

For *homogeneous* fluid, the following ellipsoidal equilibrium figures for which the entire mass *uniformly rotates as a rigid body* about a fixed axis are well known:

the Maclaurin spheroid (1742),
the Jacobi ellipsoid (1834),

In addition, in the case of a homogeneous fluid there also exist *figures of equilibrium with internal flows*:

the Dedekind ellipsoid (1861),
the Riemann ellipsoids (1861).

Remark

The discovery of the Dedekind and Riemann ellipsoids was inspired by the work of [1] where the dynamical equations for a liquid homogeneous self-gravitating ellipsoid were obtained (for this system all the above-mentioned figures of equilibrium are fixed points). For a recent review of dynamical aspects concerning liquid and gaseous self-gravitating ellipsoids and a detailed list of references, see [2]. We also note the integrability cases found in a related problem of gaseous ellipsoids [3].

[1] Dirichlet, G. L.: Untersuchungen über ein Problem der Hydrodynamik (Aus dessen Nachlass hergestellt von Herrn R. Dedekind zu Zürich). J. Reine Angew. Math. (Crelle's Journal) **58**, 181–216 (1861).

[2] Borisov, A. V., Mamaev, I. S., Kilin, A. A.: The Hamiltonian Dynamics of Self-gravitating Liquid and Gas Ellipsoids. Regul. Chaotic Dyn. **14**(2), 179–217 (2009)

[3] Gaffet, B.: Spinning gas clouds: Liouville integrability. J. Phys. A Math. Gen. **34**, 2097–2109 (2001).

While an enormous amount of research was devoted in the 19th and 20th centuries to asymmetric figures of equilibrium (see, e.g., references in [1–5]), the Maclaurin spheroid remains the most important for applications to the theory of the figures of planets. However, it is well known that for all planets of the Solar System the actual compression is different from the compression of the corresponding Maclaurin spheroid obtained from the characteristics of the planet. Usually this difference is attributed to the density stratification of the planet and it necessitates investigating inhomogeneous figures of equilibrium.

For a stratified fluid mass rotating as a rigid body with small angular velocity ω, Clairaut (1743) obtained the equation of a spheroid which is an equilibrium figure to first order in ω^2. Subsequently investigations of such figures were continued in the work of Laplace, Legendre and Lyapunov. Lyapunov obtained a final solution to this problem in the form of a power series in the small parameter ω^2 and found their convergence.

[1] Appell, P.: Traité de Mécanique Rationnelle: T. 4-1. Figures d'Équilibre d'une Masse liquide Homogène en Rotation. Gautier-Villars, Paris (1921).
[2] Liouville, J.: Sur la Figure d'une Masse Fluide Homogène, en E'quilibre et Douée d'un Mouvement de Rotation. J. de l'École Polytechnique **14**, 289–296 (1834).
[3] Lyttleton, R. A.: The Stability of Rotating Liquid Masses. Cambridge University Press, Cambridge (1953).
[4] Borisov, A. V., Mamaev, I. S., Kilin, A. A.: The Hamiltonian Dynamics of Self-gravitating Liquid and Gas Ellipsoids. Regul. Chaotic Dyn. **14**(2), 179–217 (2009).
[5] Chandrasekhar, S.: Ellipsoidal Figures of Equilibrium. Yale University Press, New Haven (1969).

On the other hand, [1, 2] and [3, Chapter 12] showed that for a stratified fluid mass rotating as a rigid body there exist no figures of equilibrium in the class of ellipsoids. We present a modern formulation of a theorem which was proven in these works:

Теорема

Suppose the body consists of a self-gravitating, ideal, stratified fluid and the density ρ is not constant along the volume. Assume that

- *the free surface of the fluid is an ellipsoid (it can be both triaxial and a spheroid),*
- *the density distribution $\rho(r)$ is such that the level surfaces $\rho(r) = \mathrm{const}$ are ellipsoids coaxial with the outer surface.*

Then such a fluid mass configuration cannot be the figure of equilibrium rotating as a rigid body about one of the principal axes.

[1] Hamy, M.: Étude sur la Figure des Corps Célestes. Ann. de l'Observatoire de Paris. Mémories **19**, 1–54 (1889).
[2] Volterra, V.: Sur la Stratification d'une Masse Fluide en Equilibre. Acta Math. **27**(1), 105–124 (1903).
[3] Pizzetti, P.: Principii della Teoria Meccanica della Figura dei Pianeti. Enrico Spoerri, Libraio-Editore, Pisa (1913).

Hamy proved this theorem for the case of a finite number of ellipsoidal layers with constant density, Volterra generalized this result to the case of continuous density distribution for a homothetic stratification of ellipsoids, and Pizzetti gave the simplest and most rigorous proof in the general case for both continuous and piecewise constant density distribution. We also mention the paper [1], which proposes higher-order corrections for finding the figures of equilibrium with stratified density. Such publications show that there is still no complete understanding regarding the equilibrium figures of celestial bodies with stratified density. We also note that [2] also attempted to prove this theorem for the case of continuous density distribution but made some errors.

[1] Kong, D., Zhang, K., Schubert, G.: Shapes of Two-Layer Models of Rotating Planets. J. Geophys. Res. **115**(E12), doi:10.1029/2010JE003720 (2010).

[2] Véronnet A.: Rotation de l'Ellipsoide Hétérogène et Figure Exacte de la Terre. J. Math. Pures et Appl., Sér. 6 **8**, 331–463 (1912.)

If one admits the possibility that the angular velocity of fluid particles is not constant for the entire fluid mass, then equilibrium figures for an arbitrary axisymmetric form of the surface and density stratification [1, Chapter 9] are possible. For example, [2] explicitly showed a spheroidal equilibrium figure with a nonuniform distribution of angular velocities for the case of homothetic density stratification. It turns out that the surfaces with equal density $\rho(r) = \mathrm{const.}$ do not coincide with the surfaces of equal angular velocity $\omega(r) = \mathrm{const.}$ S. A. Chaplygin tried to use the resulting solution to explain the dependence of the angular velocity of rotation of the outer layers of the Sun on the latitude.
In [3] an explicit solution of another kind was found for which the equilibrium figure is a spheroid consisting of two fluid masses of different density $\rho_1 \neq \rho_2$ separated by the spheroidal boundary confocal to the outer surface, with each layer rotating at constant angular velocity such that $\omega_1 \neq \omega_2$. A generalization of this solution to the case of an arbitrary finite number of "confocal layers" was obtained by [4].

[1] Pizzetti, P.: Principii della Teoria Meccanica della Figura dei Pianeti. Enrico Spoerri, Libraio-Editore, Pisa (1913).
[2] Chaplygin, S. A.: Steady-State Rotation of a Liquid homogeneous spheroid In Collected works: Vol. 2. Hydrodynamics. Aerodynamics. Gostekhizdat, Moscow (1948).
[3] Montalvo, D., Martínez, F. J., Cisneros, J.: On Equilibrium Figures of Ideal Fluids in the Form of Confocal Spheroids Rotating with Common and Different Angular Velocities. (1982).
[4] Esteban, E. P., Vasquez, S.: Rotating Stratified Heterogeneous Oblate Spheroid in Newtonian Physics. Celestial Mech. Dynam. Astronom. **81**(4), 299–312 (2001).

In this paper we obtain a generalization of this solution to the case of an arbitrary confocal (both continuous and piecewise constant) density stratification. For comparison, we also present Chaplygin's solution for the homothetic stratification. In addition, we show that in the case of a space with constant curvature the homogeneous (curvilinear) spheroid is a figure of equilibrium only under the condition of a nonuniform distribution of the angular velocities of fluid particles $\omega(r) \neq \mathrm{const}$. In this case the solution can be represented as a power series in the space curvature.

Equations of motion and axisymmetric equilibrium figures

In this case, to solve specific problems, it is convenient to use special curvilinear (nonorthogonal) coordinates, which we denote by $\boldsymbol{q} = (q_1, q_2, q_3)$. Therefore, we first represent the equations describing this system in an appropriate form.
Suppose that an element of the fluid has coordinates $\boldsymbol{q}$ at a given time t. Let $\dot{\boldsymbol{q}} = (\dot{q}_1, \dot{q}_2, \dot{q}_3)$ denote the rates of change of its coordinates during the motion. They depend on both the coordinates $\boldsymbol{q}$ of the chosen element and time t: $\dot{q}_i = \dot{q}_i(\boldsymbol{q}, t)$ and the total derivative of any function f of $\boldsymbol{q}$, and t is calculated from the formula

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial q_i} \dot{q}_i. \quad (49)$$

Let $\mathsf{G} = \|g_{ij}\|$ denote the metric tensor corresponding to these coordinates. In the case of orthogonal coordinates $\mathsf{G} = \mathrm{diag}(h_1^2, h_2^2, h_3^2)$, where h_i are the Lamé coordinates.
As is well known [1], the equations of motion for a fluid in a potential field can be represented as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} - \frac{1}{\rho}\frac{\partial p}{\partial q_i}, \quad (50)$$

where ρ is the density, p is the pressure, U is the specific potential of external forces, and T is the specific kinetic energy of the fluid calculated from the formula $T = \frac{1}{2}\sum_{i,j} g_{ij}\dot{q}_i\dot{q}_j$.

[1] Kochin, N. E., Kibel, I. A., Rose, N. V. Theoretical Hydromechanics (in Russian), Vol. 1. Fizmatgiz, Moscow (1963).

The continuity equation written in this notation become

$$\frac{\partial \rho}{\partial t} + \frac{1}{g}\sum_i \frac{\partial}{\partial q_i}(\rho g \dot{q}_i) = 0, \quad g = \sqrt{\det \mathsf{G}}. \tag{51}$$

In the case of a self-gravitating fluid the gravitational potential $U(\boldsymbol{q},t)$ can be calculated from the equation

$$\Delta U = 4\pi G \rho(\boldsymbol{q},t), \tag{52}$$

where G is the universal gravitational constant and the Laplacian is given by the well-known relation

$$\Delta = \frac{1}{g}\sum \frac{\partial}{\partial q_i}\left(g g^{ij}\frac{\partial}{\partial q_j}\right), \quad \|g^{ij}\| = \mathsf{G}^{-1},$$

assuming that outside the liquid body the density vanishes: $\rho = 0$.
In the absence of external influences at the free boundary ∂B of the fluid mass the pressure vanishes:

$$p\big|_{\partial B} = 0,$$

and the gravitational potential and its normal derivative are continuous:

$$U_{\text{in}}\big|_{\partial B} = U_{\text{out}}\big|_{\partial B}, \quad \left.\frac{\partial U_{\text{in}}}{\partial n}\right|_{\partial B} = \left.\frac{\partial U_{\text{out}}}{\partial n}\right|_{\partial B}, \tag{53}$$

where the indices in and out denote the quantities inside and outside the body, respectively (note that not only normal, but all first derivatives are continuous, even if the density is discontinuous).

Steady-state axisymmetric flows

To explore possible figures of equilibrium, we choose curvilinear coordinates $\boldsymbol{q} = (r, \mu, \varphi)$, which are related to the Cartesian coordinates as follows

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = Z(r, \mu).$$

Here the function $Z(r, \mu)$ is chosen so as to obtain a free surface of the fluid mass for one of the values $\mu = \mu_0$. Its specific form will be defined by an appropriate problem statement. The metric tensor is given by

$$\mathsf{G} = \begin{pmatrix} 1 + Z_r^2 & Z_r Z_\mu & 0 \\ Z_r Z_\mu & Z_\mu^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \quad g = \sqrt{\det \mathsf{G}} = r Z_\mu,$$

where $Z_r = \frac{\partial Z}{\partial r}$, $Z_\mu = \frac{\partial Z}{\partial \mu}$.

We shall seek a steady-state solution of (50), for which the velocity distribution has the form

$$\dot{r}=0, \quad \dot{\mu}=0, \quad \dot{\varphi}=\omega(r,\mu), \tag{54}$$

and the functions U, p, and ρ do not depend on φ. Then, substituting (54) into (50) and (52), we obtain the system of equations

$$\begin{gathered}
\frac{\partial U}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=r\omega^2, \quad \frac{\partial U}{\partial \mu}+\frac{1}{\rho}\frac{\partial p}{\partial \mu}=0, \\
\Delta_{r\mu}U=4\pi G\rho(r,\mu), \\
\Delta_{r\mu}=\frac{1}{rZ_\mu}\frac{\partial}{\partial r}\left(rZ_\mu\frac{\partial}{\partial r}\right)+\frac{1}{Z_\mu}\frac{\partial}{\partial \mu}\left(\frac{1+Z_r^2}{Z_\mu}\frac{\partial}{\partial \mu}\right)-\frac{1}{rZ_\mu}\left(\frac{\partial}{\partial r}\left(rZ_r\frac{\partial}{\partial \mu}\right)+\frac{\partial}{\partial \mu}\left(rZ_r\frac{\partial}{\partial r}\right)\right), \\
p(r,\mu)\big|_{\mu=\mu_0}=0.
\end{gathered} \tag{55}$$

Note that the continuity equation (51) holds identically.

We choose the function $Z(r,\mu)$ defining the curvilinear coordinates in such a way that all coordinate surfaces $\mu=\mathrm{const}$ are compact, and choose a value of $\mu=\mu_0$ which corresponds to the boundary of the fluid and defines the distribution of density $\rho(r,\mu)$.

Then, according to (55), after solving the equation for the potential one can always choose a distribution of pressure and of the squared angular velocity, which satisfy the first pair of equations:

$$p(r,\mu) = -\int\limits_{\mu_0}^{\mu} \rho \frac{\partial U}{\partial \mu}\, d\mu,$$

$$\omega^2(r,\mu) = \frac{1}{r\rho}\left(\rho_0 \frac{\partial U}{\partial r}(r,\mu_0) + \int\limits_{\mu_0}^{\mu}\left(\frac{\partial U}{\partial r}\frac{\partial \rho}{\partial \mu} - \frac{\partial U}{\partial \mu}\frac{\partial \rho}{\partial r}\right) d\mu\right), \quad \rho_0 = \rho(r,\mu_0).$$

A possible obstruction to the existence of such equilibrium figures is that $\omega^2(r,\mu)$, defined from these equations, may turn out to be negative. The problem of equilibrium figures becomes more nontrivial when we impose some restrictions on the distribution of angular velocity.

L. Lichtenstein and R. Wavre found sufficient conditions under which a fluid-filled region obviously possesses a plane of symmetry, see, for example, [1].

[1] Lichtenstein, L.: Gleichgewichtsfiguren Rotierender Flüssigkeiten. Springer, Berlin (1933).

Теорема

Assume that for an inhomogeneous self-gravitating mass of perfect fluid the following is satisfied:

1. *the fluid is at relative equilibrium where all particles rotate about the fixed axis Oz, and their angular velocity depends only on the distance to the axis of rotation: $\omega = \omega(r^2)$,*
2. *the density is a piecewise continuous function,*
3. *the fluid-filled region consists of a finite number of disjoint bounded (homogeneous or inhomogeneous) fluid volume whose boundaries are homeomorphic to spheres or tori.*

Then the fluid-filled region possesses a plane of symmetry perpendicular to the axis Oz.

It is also obvious that the center of mass lies on the intersection of the symmetry plane with the axis of rotation Oz.

General equations for locally nonconstant and locally constant density distributions
We now consider the case where the level surfaces of stratification of density ρ coincide with the level surfaces of angular velocity ω (i.e., the fluids of equal density move with equal angular velocity); choosing them as coordinate lines $\mu = \text{const}$, we represent this condition as

$$\rho = \rho(\mu), \quad \omega = \omega(\mu). \tag{56}$$

Eliminating the pressure from the first pair of equations of the system (55) (multiplying them by ρ and differentiating the first one with respect to μ and the second one with respect to r and subtracting one from the other), we obtain

$$\rho'(\mu)\,\frac{\partial U(r,\mu)}{\partial r} = r\big(\rho(\mu)\omega^2(\mu)\big)', \tag{57}$$

where the prime denotes the derivative with respect to μ.
Let us consider the main consequences of this equation (expressing the restrictions to the gravitational potential inside the figure), which result from the conditions of mechanical equilibrium. We see that according to (57) it is necessary to analyze two cases separately. In the first of the cases we assume that $\rho'(\mu)$ vanishes only at isolated points, in the second case we have to consider a situation where on the whole interval $\rho(\mu) \equiv 0$, $\mu \in (\mu_1, \mu_2)$. Let us consider them in succession.

1. *The case of locally nonconstant density.* If we assume that in some interval $\mu \in (\mu_1, \mu_2)$

$$\rho'(\mu) \neq 0, \quad \text{inside.}$$

then, according to (57), the potential U in this volume of fluid can be written as

$$U(r,\mu) = \frac{1}{2}u(\mu)r^2 + v(\mu). \tag{58}$$

If $\rho'(\mu_*) = 0$ at some isolated point μ_*, then on the left and right of μ_* the potential is represented in the form (58), and due to continuity of $U(r,\mu)$ the limits of the functions $u(\mu)$ and $v(\mu)$ in μ_* on the left and right coincide. In this case, if $\rho(\mu_*) \neq 0$, then the equation $\frac{d\omega^2(\mu)}{d\mu}\Big|_{\mu=\mu_*} = 0$ holds for the angular velocity ω of this layer. From the first pair of equations (55), we obtain the unknowns $p(r,\mu)$ and $\omega(\mu)$ in the form

$$\begin{gathered} p = -\tfrac{1}{2}P(\mu)r^2 - Q(\mu), \quad \omega^2(\mu) = u(\mu) - \tfrac{P(\mu)}{\rho(\mu)}, \\ P(\mu) = \int\limits_{\mu_0}^{\mu} u'(\xi)\rho(\xi)\,d\xi, \quad Q(\mu) = \int\limits_{\mu_0}^{\mu} v'(\xi)\rho(\xi)\,d\xi. \end{gathered} \tag{59}$$

Obviously,

$$p(r,\mu)\Big|_{\mu=\mu_0} = 0, \quad \frac{d\omega^2}{d\mu}\Big|_{\mu=\mu_0} = 0.$$

Hence, it follows that the figure of equilibrium of a fluid with density stratification and angular velocity of the form (56) exists if and only if there exist functions $Z(r,\mu)$ and $u(\mu)$, $v(\mu)$ satisfying the equation

$$\Delta_{r,\mu}\left(\frac{1}{2}u(\mu)r^2+v(\mu)\right)=4\pi G\rho(\mu), \tag{60}$$

and the potential inside the fluid mass has the form (58).

2. *The case of locally constant density.* We now consider a situation where in some layer the density takes a constant value:

$$\rho(\mu)=\rho_0=\mathrm{const},\quad \mu\in(\mu_1,\mu_2),$$

then, according to (57), we conclude that the angular velocity of the entire layer is also constant:

$$\omega(\mu)=\omega_0=\mathrm{const},\quad \mu\in(\mu_1,\mu_2).$$

Taking this result into account, we integrate the first pair of equations (55) and obtain the following relation for the function $U+\frac{p}{\rho_0}$ in the layer:

$$U+\frac{p}{\rho_0}=\frac{1}{2}\omega_0^2r^2+\Phi_0,\quad \Phi_0=\mathrm{const}. \tag{61}$$

Furthermore, at
all points at the boundaries of the layer $\mu = \mu_i$, $i = 1, 2$ (see Fig. 2) the pressure inside and outside must be the same:

$$p_{\mathrm{in}}(r,\mu)\Big|_{\mu=\mu_i} = p_{\mathrm{out}}(r,\mu)\Big|_{\mu=\mu_i}. \qquad (62)$$

The potential in the layer also satisfies the Laplace equation

$$\Delta_{r\mu} U_{\mathrm{in}}(r,\mu) = 4\pi G\rho_0,$$

and at the boundaries
conditions (53) hold. In principle, in the general case
an inhomogeneous figure of equilibrium can consist of parts, with both locally constant and locally nonconstant density.

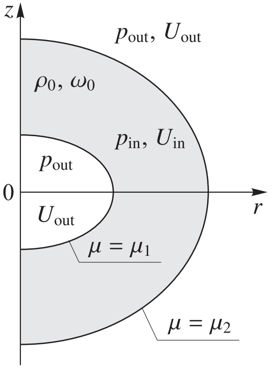


Рис. 2: Distribution with a layer on which the density takes a constant value

The family of confocal spheroids

Consider a particular case in which the sought-for solution exists. We shall show that in the case of confocal stratification of the density of a spheroid the gravitational potential is written as (58).

Choose the parameterization of confocal stratification in $\mathbb{R}^3$ as follows

$$\frac{x^2+y^2}{d^2(1+\mu^2)} + \frac{z^2}{d^2\mu^2} = 1, \quad \mu \in [0, +\infty),$$

where d is the focal distance of the meridianal section (see Fig. 3). Thus, the parameter μ defines the ratio between the small semiaxis of the spheroid and the focal distance, and the eccentricity e is expressed by the formula

$$e = \frac{1}{\sqrt{1+\mu^2}}. \tag{63}$$

Expressing z, we find

$$Z(r,\mu) = \pm\sqrt{d^2\mu^2 - r^2\frac{\mu^2}{\mu^2+1}}. \tag{64}$$

Remark

It can be shown that for a prolate spheroidal stratification $\left(i.\,e., \text{ for } \frac{r^2}{d^2\mu^2} + \frac{z^2}{d^2(\mu^2+1)} = 1\right)$ *this solution leads to a negative square of the angular velocity of rotation of the layers* $(\omega^2(\mu) < 0)$*, therefore, we will not consider it.*

If the boundary of the spheroid filled with a fluid has semiaxes a and b (see Fig. 3), then the focal distance d and the coordinate of the boundary μ_0 are defined by

$$d = \sqrt{a^2 - b^2}, \quad \mu_0 = \frac{b}{\sqrt{a^2 - b^2}}. \tag{65}$$

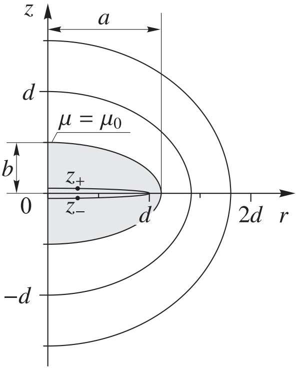


Рис. 3: Meridional sections of the surfaces $\mu = \mathrm{const}$

Proposition

The gravitational potential for a spheroid with confocal stratification has the form

$$U = \frac{k}{2}\left(\frac{1}{2}\frac{r^2\widetilde{u}(\mu)}{1+\mu^2} + d^2\widetilde{v}(\mu)\right), \; k = 4\pi G. \tag{66}$$

For the internal points

$$\begin{aligned} \widetilde{u}^{\,\text{in}} &= I_0(\mu)((1+3\mu^2)\,\text{arcctg}(\mu) - 3\mu) - I_1(\mu)(1+3\mu^2) \\ \widetilde{v}^{\,\text{in}} &= -I_0(\mu)((1+\mu^2)\,\text{arcctg}(\mu) - \mu) + I_1(\mu)(1+\mu^2) + 2I_2(\mu) \end{aligned} \tag{67}$$

$$I_0(\mu) = \int_0^{\mu} \rho(\xi)(1+3\xi^2)\,d\xi, \quad I_1(\mu) = \int_{\mu_0}^{\mu} \rho(\xi)((1+3\xi^2)\,\text{arcctg}(\xi) - 3\xi)\,d\xi,$$

$$I_2(\mu) = \int_{\mu_0}^{\mu} \xi\rho(\xi)\,d\xi.$$

For the external points

$$\widetilde{u}^{\,\text{out}} = I_0(\mu_0)((1+3\mu^2)\,\text{arcctg}(\mu) - 3\mu), \quad \widetilde{v}^{\,\text{out}} = I_0(\mu_0)(\mu - (1+\mu^2)\,\text{arcctg}(\mu)). \tag{68}$$

We note that no assumptions about differentiability of the distribution of $\rho(\mu)$ were used in proving this proposition. Therefore, this result holds for any (including piecewise constant) distribution for which the integrals (67) converge.
Now, using the representations (67) and (66) for the potential inside the figure, we can find $\omega(\mu)$ and $p(r,\mu)$ from the equilibrium conditions (see the first pair of equations in (55)). Finally, integrating by parts, we obtain the following theorem.

Theorem

Suppose that an inhomogeneous perfect fluid fills an oblate spheroid with the semiaxes $a_0 = d(1+\mu_0^2)^{\frac{1}{2}}$, $b_0 = d\mu_0$ and that the surfaces of constant density of the fluid coincide with the family of confocal ellipsoidal surfaces with the semiaxes $a = d(1+\mu^2)^{\frac{1}{2}}$ and $b = d\mu$, $\mu \in [0,\mu_0]$. Then, for each density distribution $\rho\colon [0,\mu_0] \to R$, there is a stationary motion such that each ellipsoidal surface of the family moves as a rigid body around the common axes with the angular velocity

$$\frac{\omega(\mu)^2}{2\pi G} = I_0(\mu_0)\frac{\rho(\mu_0)}{\rho(\mu)}\frac{(1+3\mu_0^2)\operatorname{arcctg}(\mu_0)-3\mu_0}{1+\mu_0^2} - \\ -\frac{2}{\rho(\mu)}\int_{\mu}^{\mu_0}\rho'(\xi)\frac{I_0(\xi)((1+3\xi^2)\operatorname{arcctg}(\xi)-3\xi)-I_1(\xi)(1+3\xi^2)}{1+\xi^2}\,d\xi, \tag{69}$$

and with the pressure distribution

$$p(r,\mu) = 4\pi G\int_{\mu_0}^{\mu}\rho(\xi)\left(\frac{r^2}{(1+\xi^2)^2}-d^2\right)(I_0(\xi)(1-\xi\operatorname{arcctg}(\xi))+I_1(\xi)\xi)d\xi. \tag{70}$$

The homogeneous Maclaurin spheroid

Let the density be constant everywhere inside some spheroid:

$$\rho(\mu) = \begin{cases} 0, & \mu_0 < \mu, \\ \rho_0, & 0 < \mu \le \mu_0, \end{cases}$$

where μ_0 is defined by (65). In this case we find the gravitational potential from Proposition 1. Inside the spheroid it can be represented as

$$U = 2\pi G\left(\frac{1}{2}\frac{r^2\widetilde{u}^{\text{in}}(\mu)}{1+\mu^2} + d^2\widetilde{v}^{\text{in}}(\mu)\right),$$

$$u^{\text{in}}(\mu) = \rho_0\big(\mu_0(1+3\mu^2)((1+\mu_0^2)\operatorname{arcctg}\mu_0 - \mu_0) - 2\mu^2\big),$$

$$v^{\text{in}}(\mu) = \rho_0(1+\mu_0^2)\big(\mu^2 - \mu_0(1+\mu^2)\operatorname{arcctg}\mu_0\big).$$

Comparing this expression with (66) and (68) as a consequence of this representation of the potential, we obtain the well-known Maclaurin theorem [1] in the case of a spheroid.

[1] Chandrasekhar, S.: Ellipsoidal Figures of Equilibrium. Yale University Press, New Haven (1969).

Theorem

The gravitational potential that is produced by an inhomogeneous spheroid with confocal stratification and density $\rho(\mu)$ at the external point is the same as the potential of a homogeneous spheroid with the density

$$\rho_0 = \langle \rho \rangle = \frac{1}{\mu_0(1+\mu_0^2)} \int\limits_0^{\mu_0} (1+3\xi^2)\rho(\xi)\,d\xi. \tag{71}$$

Next, from (69) and taking into account the relationship (63) between μ_0 and the eccentricity, we obtain the well-known expression for the angular velocity ω_0 of the Maclaurin spheroid

$$\frac{\omega_0^2}{2\pi G\rho_0} = \mu_0\Big((1+3\mu_0^2)\operatorname{arcctg}\mu_0 - 3\mu_0\Big) = \frac{\sqrt{1-e^2}}{e^3}\Big((3-2e^2)\arcsin e - 3e\sqrt{1-e^2}\Big). \tag{72}$$

Using (70), we find the pressure for the Maclaurin spheroid:

$$\frac{p}{2\pi G\rho_0^2} = \frac{(\mu_0^2-\mu^2)(1-\mu_0\operatorname{arcctg}\mu_0)}{1+\mu^2}\big(d^2(1+\mu^2)(1+\mu_0^2) - r^2\big). \tag{73}$$

It can be shown that the level surfaces (73) are homothetic spheroids. Using (64) and a relation defining the homothetic stratification, which in our case takes the form

$$\frac{r^2}{d^2(1+\mu_0^2)} + \frac{z^2}{d^2\mu_0^2} = m,$$

we find

$$r = \frac{d^2(1+\mu_0^2)(1+\mu^2)(m\mu_0^2-\mu^2)}{\mu^2-\mu_0^2}.$$

Then, substituting r into (73), we obtain

$$\frac{p}{2\pi G\rho_0^2} = d^2\mu_0^2(1+\mu_0^2)(1-\mu_0 \operatorname{arcctg}\mu_0)(1-m).$$

If we compare the expressions (72) and (69) for $\mu = \mu_0$, then we obtain the following result:

Theorem

For an arbitrary confocal stratification the angular velocity on the outer surface of the inhomogeneous spheroid is the same as the angular velocity ω_0 of the Maclaurin spheroid with density $\rho_0 = \langle\rho\rangle$:

$$\frac{\omega_0^2}{2\pi G\langle\rho\rangle} = \mu_0((1+3\mu_0^2)\operatorname{arcctg}(\mu_0) - 3\mu_0), \tag{74}$$

where $\langle\rho\rangle$ is the average density of the spheroid (71).

Thus, we see that for any confocal stratification the rotation of the body (planet) does not differ visually from from that of the Maclaurin spheroid.

The Chaplygin problem — a spheroid with homothetic density distribution

As is well known,
the homothetic stratification is given by

$$\frac{z^2}{b^2} + \frac{r^2}{a^2} = \sigma, \quad \sigma \in [0, +\infty),$$

where, assuming that a and b are the principal semiaxes of a spheroid filled with a fluid (see Fig. 4), we obtain

$$\sigma_0 = 1, \quad Z(r, \sigma) = \pm b\sqrt{\sigma - \frac{r^2}{a^2}}.$$

Again we set

$$\rho = \begin{cases} \rho(\sigma) \text{ (does not depend on } r\text{)}, & \sigma \le 1, \\ 0, & \sigma > 1. \end{cases}$$

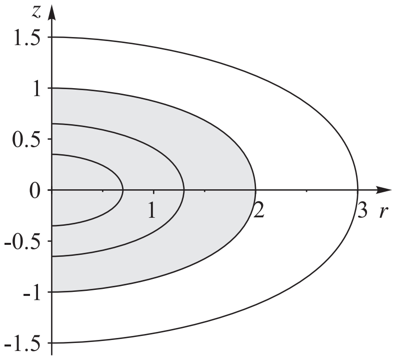


Рис. 4: Meridional sections of the surfaces $\sigma = \text{const}$ with homothetic stratification

Using the second of Eqs. (55) and noting that $p\big|_{\sigma=1} = 0$, we obtain the pressure, which can be represented as

$$p(r,\sigma) = \rho_1 U(r,1) - \rho(\sigma)U(r,\sigma) + \int_1^{\sigma} U(r,\sigma)\,\frac{\partial \rho}{\partial \sigma}\,d\sigma, \quad \rho_1 = \rho(1).$$

In a similar manner, substituting the pressure from the first of Eqs. (55), we obtain

$$\omega^2(r,\sigma) = \frac{1}{r\rho(\sigma)}\left(\rho_1 \frac{\partial U}{\partial r}(r,1) + \int_1^{\sigma} \frac{\partial U}{\partial r}(r,\sigma)\frac{\partial \rho}{\partial \sigma}\,d\sigma\right). \tag{75}$$

Thus, to complete the solution, we only need to find the potential from the equation

$$\Delta_{r,\sigma} U(r,\sigma) = 4\pi G\rho(\sigma).$$

In [1] a convenient integral representation of the potential for a (three-axial) ellipsoid with homothetic density stratification is obtained. Applying it to the case of the spheroid $\sigma = 1$ gives

$$U^{\text{in}}(r,z) = \pi G a^2 b^2 \int_0^{\infty} \frac{f(1) - f\left(\frac{r^2}{a^2+s} + \frac{z^2}{b^2+s}\right)}{\Delta(s)}\, ds,$$

$$U^{\text{out}}(r,z) = \pi G a^2 b^2 \int_{s_0}^{\infty} \frac{f(1) - f\left(\frac{r^2}{a^2+s} + \frac{z^2}{b^2+s}\right)}{\Delta(s)}\, ds, \qquad (76)$$

$$\Delta(s) = (a^2 + s)\sqrt{b^2 + s},$$

where the function $f(\sigma)$ is related with the density of the fluid by

$$\rho(\sigma) = \frac{df(\sigma)}{d\sigma},$$

and the quantity s_0 for given (r,z), which correspond to a point outside the liquid spheroid, is defined as the root of the equation

$$\frac{r^2}{a^2+s_0} + \frac{z^2}{b^2+s_0} = 1.$$

[1] Ferrers, N. M.: On the Potentials, Ellipsoids, Ellipsoidal Shells, Elliptic Laminae, and Elliptic Rings, of Variable Densities. Quart. J. Pure Appl. Math. **14**, 1–23 (1875).

As an example, we consider the density distribution of the form

$$\rho(\sigma) = \rho_0(1 - \alpha\sigma^n), \quad n = 1, 2, 3. \tag{77}$$

Given the average density $\langle\rho\rangle$ of the body and the ratio between the densities at the center and on the surface $\eta = \frac{\rho_0}{\rho_1}$, we now define the constants ρ_0 and α:

$$\alpha = \frac{\eta - 1}{\eta}, \quad \rho_0 = \frac{\eta(3 + 2n)\langle\rho\rangle}{3 + 2n\eta}. \tag{78}$$

Set

$$\eta = 5, \quad \frac{b}{a} = \frac{1}{2}.$$

Further, we find the potential from (76) and obtain the angular velocity from (75). The meridional sections of the surfaces $\frac{\omega^2}{2\pi G\langle\rho\rangle} = \mathrm{const}$ with equal spacings for different $n = 1, 2, 3$ are shown in Fig. 5. The graphs of change in the relation $\frac{\omega^2}{2\pi G\langle\rho\rangle}$ along the small semiaxis b is shown in Fig. 6.

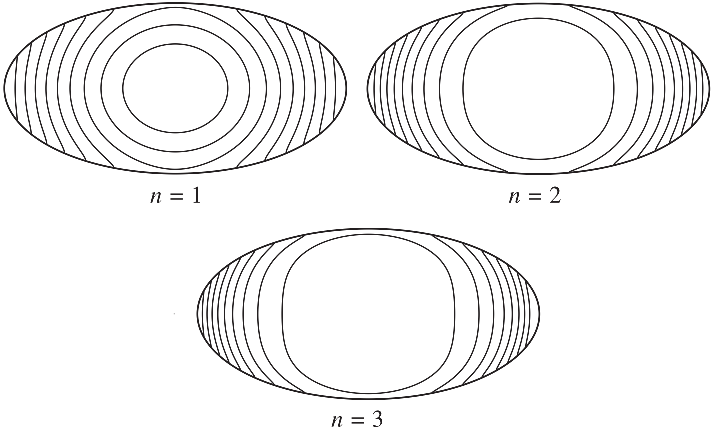


Рис. 5: Meridional sections of the surfaces $\frac{\omega^2}{2\pi G\langle\rho\rangle} = \text{const}$ with equal spacings

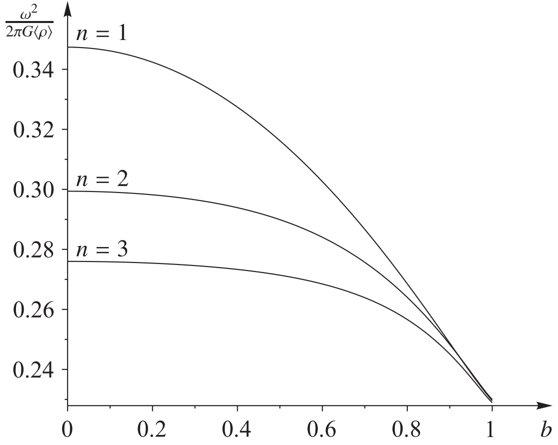


Рис. 6: The change of $\frac{\omega^2}{2\pi G\langle\rho\rangle}$ along the small semiaxis b for different n

For the densities from Figs. 5 and 6 one can draw the following conclusions:
1. *The closer the center of the spheroid, the slower the change in the angular velocity.*
2. *For $n = 1$ the level surfaces near the center of the spheroid are concentric spheres. Further, as n increases, the region in which the level lines are closed surfaces increases. For $n > 1$ these closed surfaces are no longer surfaces of the second order.*
Let us consider in more detail the angular velocity at the boundary of the spheroid at the equator with densities of the form (77), but now with an arbitrary n. From (75), changing the variable $s = a^2(t-1)$, we obtain the angular velocity on the surface:

$$\frac{\omega_n^2(r,1)}{2\pi G} = \rho_0 e^2 \sqrt{1-e^2} \int_1^\infty \frac{t-1}{t^2(t-e^2)^{3/2}} \left(1 - \frac{\alpha t^{-n}}{(t-e^2)^n} \left((t-1)e^2 \frac{r^2}{a^2} + t(1-e^2)\right)^n\right) dt,$$

that is, for $r = a$ we have

$$\frac{\omega_n^2(a,1)}{2\pi G} = \rho_0 e^2 \sqrt{1-e^2} \int_1^\infty \frac{(t-1)(1-\alpha t^{-n})}{t^2(t-e^2)^{3/2}} dt.$$

Explicitly integrating gives

$$\frac{\omega_n^2(a,1)}{2\pi G} = \rho_0\omega_m^2 + \frac{2\alpha\rho_0 e^2}{3+2n}\left(\frac{\sqrt{1-e^2)}(2n(1-e^2)+3-2e^2)}{5+2n}F\left(\frac{3}{2}, n+\frac{5}{2}, n+\frac{7}{2}, e^2\right) - 1\right), \tag{79}$$

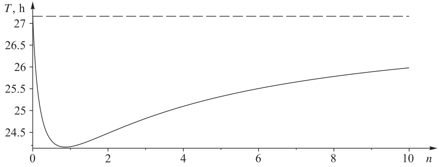
where ω_m^2 is the dimensionless angular velocity of the Maclaurin spheroid:

$$\omega_m^2 = \frac{\sqrt{1-e^2}}{e^3}\left((3-2e^2)\arcsin e - 3e\sqrt{1-e^2}\right).$$

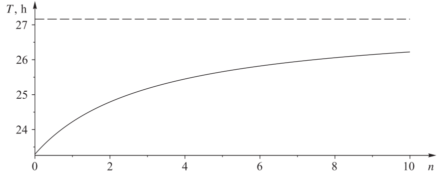
Substituting the expression (79) into the relation for the angular velocity, we obtain for two values of n

$$\frac{\omega_0^2(a,1)}{2\pi G\rho_0(1-\alpha)} = \frac{\omega_\infty^2(a,1)}{2\pi G\rho_0} = \omega_m^2.$$

Further, we shall define ρ_0 and α from various known data for the Earth:
We are given the average density of the body $\langle\rho\rangle = 5.51$ *g/cm*^{3} *and the ratio between the densities on the surface and at the center* $\frac{\rho_0}{\rho_1} = 5$. In this case ρ_0 and α are defined by (79), and the dependence of the period of revolution at the equator T on n is shown in Fig. 7(a).



(a)



(b)

Рис. 7: The dependence of period T on n at the equator (a) for $\langle\rho\rangle = 5.51$ g/cm^3 and $\frac{\rho_0}{\rho_1} = 5$; (b) for $\langle\rho\rangle = 5.51$g/cm^3 and $\varepsilon = 2.5$

As can be seen in Fig. 7(a), $T(n)$ reaches the minimum at the point $T(0.8675) = 24.1610$ h.

We are given the average density of the body, $\langle\rho\rangle = 5.51$ g/cm^3, and the ratio between the density on the surface and the average density, $\frac{\rho_1}{\langle\rho\rangle} = \varepsilon = 2.5$. The dependence of the period of revolution at the equator T on n is shown in Fig. 7(b).

The dependence of the period of revolution T on the polar radius r on the surface is shown in Fig. 8.

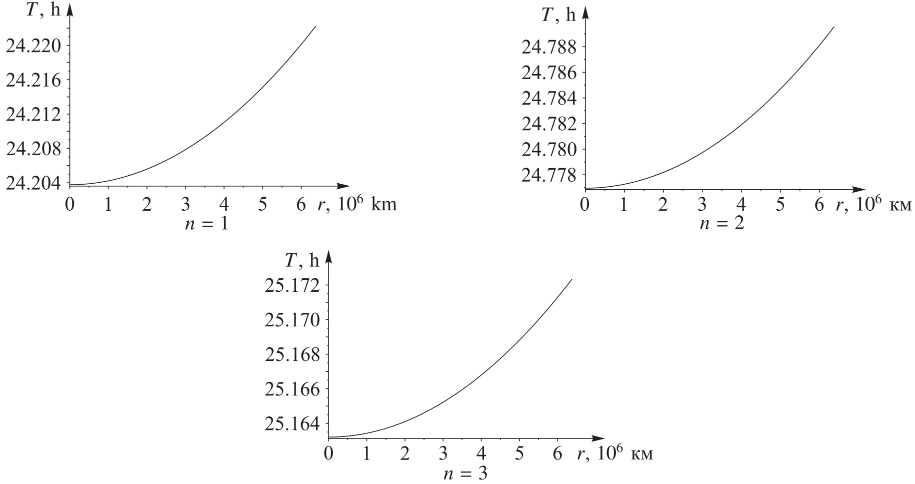


Рис. 8: The dependence of period T on the polar radius on the surface of the inhomogeneous spheroid $\langle\rho\rangle = 5.51\text{g/cm}^3$ and $\varepsilon = 2.16$ for $n = 1$, $n = 2$ and $n = 3$.

Figures of equilibrium in S^3

One of the generalizations of the above results is that they are carried over to the spaces of constant curvature S^3 and L^3, by analogy with celestial mechanics of point masses [1–5]. There is a vast classical and recent literature on the dynamics of gravitating point masses (see [6–10]), in which, for example, the well-known analogs of the Kepler law and those of the three-body problem were studied.

[1] Borisov, A. V., Mamaev, I. S.: Poisson Structures and Lie Algebras in Hamiltonian Mechanics. Izd. UdSU, Izhevsk (1999) (in Russian).
[2] Borisov, A. V., Mamaev, I. S.: The Restricted Two-Body Problem in Constant Curvature Spaces. Celestial Mech. Dynam. Astronom. **96**(1), 1–17 (2006).
[3] Killing, H. W.: Die Mechanik in den Nichteuklidischen Raumformen. J. Reine Angew. Math. **XCVIII**(1), 1–48 (1885).
[4] Kozlov, V. V., Harin, A. O.: Kepler's Problem in Constant Curvature Spaces. Celestial Mech. Dynam. Astronom. **54**(4) 393–399 (1992).
[5] Schrödinger, E.: A Method of Determining Quantum-Mechanical Eigenvalues and Eigenfunctions. Proc. Roy. Irish Acad. Sect. A **46**, 9–16 (1940).
[6] Albouy, A.: There is a Projective Dynamics. Eur. Math. Soc. Newsl. (89), 37–43 (2013).
[7] Borisov, A. V., Mamaev, I. S., Kilin, A. A.: Two-Body Problem on a Sphere. Reduction, Stochasticity, Periodic Orbits. Regul. Chaotic Dyn. **9**(3), 265–279 (2004).
[8] Borisov, A. V., Mamaev, I. S.: The Restricted Two-Body Problem in Constant Curvature Spaces. Celestial Mech. Dynam. Astronom. **96**(1), 1–17 (2006).
[9] Borisov, A. V., Mamaev, I. S.: Relations Between Integrable Systems in Plane and Curved Spaces. Celestial Mech. Dynam. Astronom. **99**(4), 253–260 (2007).
[10] Bizyaev, I. A., Borisov, A. V., Mamaev, I. S.: Figures of equilibrium of an inhomogeneous self-gravitating fluid. Nonlinear Dynamics. **10**(1), 73–100 (2014) (in Russian).

However, a particular generalization of the theorems of Newtonian potential to S^3 and L^3 was performed only in [1]. As will be shown below, in this case the problem of equilibrium figures becomes considerably more complex. In particular, even in the case of homogeneous ellipsoids the rigid body rotation of a fluid mass is impossible (we recall that an ellipsoid in curved space is said to be a body obtained by the intersection of the sphere S^3 or the Lobachevsky space L^3, embedded in $\mathbb{R}^4$, with a conical quadric). One of the difficulties is due to the fact that although some generalizations of Ivory's theorem on the potential of the elliptic layer [1] are possible, this and similar theorems cannot be completely extended to S^3 and L^3 (they are closely related to the homogeneity of plane space).

Remark

Generalizations of the problem of equilibrium figures to the relativistic case are also possible, see, e.g., the review [2]. Unfortunately, attempts to obtain explicit analytical exact solutions along these lines have yielded no results so far. This direction is a new research area.

[1] Kozlov, V. V.: The Newton and Ivory Theorems of Attraction in Spaces of Constant Curvature. (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. (5), 43–47 (2000).

[2] Meinel, R., Ansorg, M., Kleinwachter, A., Neugebauer, G., Petroff, D.: Relativistic Figures of Equilibrium. Cambridge University Press, Cambridge (2008).

Steady-state axisymmetric solutions in S^3

To explore possible figures of equilibrium in S^3, we choose curvilinear coordinates, as was done for the plane space E^3. For convenience, we assume S^3 to be embedded into E^4, then the transition to the coordinates under consideration has the form

$$x_0 = \pm\sqrt{R^2 - r^2 - Z^2(r,\mu)},\ x_1 = Z(r,\mu),\ x_2 = r\cos(\varphi),\ x_3 = r\sin(\varphi),$$

where $Z(r,\mu)$ is defined, as before, by the specific problem statement. The metric tensor can be represented as

$$\mathbf{G} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & r^2 \end{pmatrix},$$

where

$$g_{11} = 1 - Z_r^2 + \frac{(r + ZZ_r)^2}{R^2 - r^2 - Z^2}, \quad g_{12} = \frac{Z_\mu(rZ + (R^2 - r^2)Z_r)}{R^2 - r^2 - Z^2}, \quad g_{22} = \frac{(R^2 - r^2)Z_\mu^2}{R^2 - r^2 - Z^2}.$$

We shall seek a steady-state solution for which the velocity distribution of fluid particles has the form

$$\dot{r} = 0, \quad \dot{\mu} = 0, \quad \dot{\varphi} = \omega(r,\mu).$$

As above, assuming that the density depends only on μ and using the equations of Section 2.1, we obtain the system

$$\frac{\partial U}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = r\omega^2, \quad \frac{\partial U}{\partial \mu} + \frac{1}{\rho}\frac{\partial p}{\partial \mu} = 0,$$

$$\triangle_{r\mu} U = 4\pi G\rho(\mu),$$

$$\triangle_{r\mu} = \frac{x_0}{rZ_\mu}\frac{\partial}{\partial r}\left[\frac{rZ_\mu}{x_0}\left(1-\frac{r^2}{R^2}\right)\frac{\partial}{\partial r}\right] + \frac{x_0}{Z_\mu}\frac{\partial}{\partial \mu}\left[\frac{1}{x_0 Z_\mu}\left(1+Z_r^2-\frac{(Z-rZ_r)^2}{R^2}\right)\frac{\partial}{\partial \mu}\right] + \tag{80}$$

$$+\frac{x_0}{rZ_\mu}\left(\frac{\partial}{\partial r}\left[\frac{r}{x_0}\left(Z_r+\frac{r(Z-rZ_r)}{R^2}\right)\frac{\partial}{\partial \mu}\right] + \frac{\partial}{\partial r}\left[\frac{r}{x_0}\left(Z_r+\frac{r(Z-rZ_r)}{R^2}\right)\frac{\partial}{\partial \mu}\right]\right),$$

where $x_0 = \sqrt{R^2 - r^2 - Z^2(r,\mu)}$ and it is assumed that the density $\rho(\mu)$ vanishes everywhere outside the body ($\mu_0 < \mu$), and at the free boundary $\mu = \mu_0$ the pressure is zero as well:

$$p(r,\mu)|_{\mu=\mu_0} = 0.$$

As we can see, the hydrodynamical equations remain the same as in E^3. Therefore, their solution inside the region ($\mu \le \mu_0$) filled with fluid can be represented as

$$\begin{gathered} p(r,\mu) = \rho_0 U(r,\mu_0) - \rho(\mu) U(r,\mu) + \int_{\mu_0}^{\mu} U(r,\mu)\, \frac{d\rho(\mu)}{d\mu}\, d\mu, \quad \rho_0 = \rho(\mu_0), \\ \omega^2(r,\mu) = \frac{1}{r\rho(\mu)} \left(\rho_0 \frac{dU}{dr}(r,\mu_0) + \int_{\mu_0}^{\mu} \frac{dU}{dr}(r,\mu)\, \frac{d\rho(\mu)}{d\mu}\, d\mu \right). \end{gathered} \tag{81}$$

A homogeneous spheroid in S^3

We now consider in more detail the case of a homogeneous spheroid, when, for $\mu \leq \mu_0$, the density $\rho(\mu) = \rho_0 = \mathrm{const}$. The generalization of confocal stratification in S^3 is given as follows:

$$\frac{x_0^2}{R^2 - d^2\mu^2} - \frac{x_1^2}{d^2\mu^2} - \frac{x_2^2 + x_3^2}{d^2(1+\mu^2)} = 0, \quad \mu \in \left[0, \frac{R}{d}\right].$$

Hence, we obtain

$$Z(r,\mu) = \pm\sqrt{d^2\mu^2 - r^2\frac{R^2+d^2}{R^2}\frac{\mu^2}{1+\mu^2}}.$$

As in the previous case, the parameter d and the boundary μ_0 of a liquid spheroid with the semiaxes a and b are given by

$$d = \sqrt{a^2 - b^2}, \quad \mu_0 = \frac{b}{\sqrt{a^2-b^2}}.$$

According to (81), in the case of a homogeneous spheroid $\frac{d\rho}{d\mu} = 0$, therefore, the angular velocity of the fluid depends only on r:

$$\omega^2(r) = \frac{1}{r}\frac{\partial U}{\partial r}(r,\mu_0). \tag{82}$$

We shall seek solutions to the equation for the potential (80) in the form of a power series in the parameter $\frac{d^2}{R^2}$:

$$U(r,\mu) = 2\pi G d^2 \sum_{n=0}^{\infty} \left(\frac{d}{R}\right)^{2n} U_n(r,\mu).$$

As can be shown, all terms of this series are polynomials in r. It is convenient to represent them as

$$U_n(r,\mu) = \sum_{n=0}^{\infty} \left(\frac{r}{d}\right)^{2m} \frac{u_{n,\mu}(\mu)}{2^m(1+\mu^2)^m}.$$

The potential $U_0(r,\mu)$ is equal (up to a multiplier) to the potential of the Maclaurin spheroid (see Section 65):

$$U_0(r,\mu) = u_{0,0}(\mu) + \frac{r^2}{d^2} \frac{u_{0,1}(\mu)}{2(1+\mu^2)},$$

inside the spheroid ($\mu \le \mu_0$):

$$u^{\text{in}}_{0,0}(\mu) = \rho_0(1+\mu_0^2)\big(\mu^2 - \mu_0(1+\mu^2)\operatorname{arcctg}\mu_0\big),$$

$$u^{\text{in}}_{0,1}(\mu) = \rho_0\Big(\mu_0(1+3\mu^2)\big((1+\mu_0^2)\operatorname{arcctg}\mu_0 - \mu_0\big) - 2\mu^2\Big),$$

outside the spheroid ($\mu_0 < \mu$):

$$u^{\text{out}}_{0,0}(\mu) = \rho_0\mu_0(1+\mu_0^2)\big(\mu - (1+\mu^2)\operatorname{arcctg}\mu\big),$$

$$u^{\text{out}}_{0,1}(\mu) = \rho_0\mu_0(1+\mu_0^2)\big((1+3\mu^2)\operatorname{arcctg}\mu - 3\mu\big).$$

We shall assume that the space curvature is very small ($R^2 \gg a^2$) and, therefore, restrict ourselves to calculating the first correction

$$U_1(r,\mu) = \frac{r^4}{d^4}\frac{u_{1,2}(\mu)}{4(1+\mu^2)^2} + \frac{r^2}{d^2}\frac{u_{1,1}(\mu)}{2(1+\mu^2)} + u_{1,0}(\mu),$$

where the functions $u_{1,0}(\mu)$, $u_{1,1}(\mu)$, and $u_{1,2}(\mu)$ satisfy the equations

$$\frac{d}{d\mu}\left((1+\mu^2)\frac{du_{1,2}}{d\mu}\right) - 20u_{1,2} + 16u_{0,1} = 0,$$
$$\frac{d}{d\mu}\left((1+\mu^2)\frac{du_{1,1}}{d\mu}\right) - 6u_{1,1} - \mu(1+\mu^2)\frac{du_{0,1}}{d\mu} - 6(2+\mu^2)u_{0,1} + 8u_{1,2} + 4\rho_0(1+\mu^2) = 0,$$
$$\frac{d}{d\mu}\left((1+\mu^2)\frac{du_{1,0}}{d\mu}\right) - 2u_{1,1} - \mu(1+\mu^2)\frac{du_{0,0}}{d\mu} + 2\mu^2(u_{0,1} + \rho_0(1+\mu^2)) = 0. \tag{83}$$

The functions $u_{1,0}$, $u_{1,1}$, and $u_{1,2}$ must also satisfy the following boundary conditions:

$$\left.\frac{du^{\text{in}}_{1,m}}{d\mu}\right|_{\mu=0} = 0, \quad m = 0,1,2.$$
$$u^{\text{in}}_{1,m}|_{\mu=\mu_0} = u^{\text{out}}_{1,m}|_{\mu=\mu_0}, \quad \left.\frac{du^{\text{in}}_{1,m}}{d\mu}\right|_{\mu=\mu_0} = \left.\frac{du^{\text{out}}_{1,m}}{d\mu}\right|_{\mu=\mu_0}, \quad m = 0,1,2.$$
$$U_1(r,\mu)\big|_{\mu=\frac{R}{d}} = O(R^2).$$

Since the solution of the resulting system is rather unwieldy, we omit it here and confine ourselves to the expression for the angular velocity of the fluid, for which, according to (82), we find

$$\frac{\omega^2(r)}{2\pi G} = \frac{u^{\text{in}}_{0,1}(\mu_0)}{1+\mu_0^2} + \frac{1}{R^2}\left(\frac{u^{\text{in}}_{1,2}(\mu_0)}{(1+\mu_0^2)^2}r^2 + \frac{u^{\text{in}}_{1,1}(\mu_0)}{1+\mu_0^2}d^2\right) + O\left(\frac{d^4}{R^4}\right).$$

Substituting the solution for $u^{\text{in}}_{1,m}(\mu_0)$ and expressing μ_0 in terms of the eccentricity of the boundary using the formula $e = \frac{1}{\sqrt{1+\mu_0^2}}$ and $d^2 = a^2 - b^2$, we obtain an explicit representation for the angular velocity in the form

$$\frac{\omega^2(r)}{2\pi G\rho_0} = \omega_{00} + \frac{1}{R^2}\left(\omega_{11}r^2 + \omega_{10}a^2\right) + O\left(\frac{d^2}{R^4}\right),$$

$$\omega_{00} = -\frac{\sqrt{1-e^2}}{e}\left(2 - \frac{3}{e^2}\right)\arcsin e - \frac{3}{e^2}(1-e^2),$$

$$\omega_{11} = -\frac{\sqrt{1-e^2}}{e}\left(12 - \frac{30}{e^2} + \frac{35}{2e^4}\right)\arcsin e + \left(\frac{4}{3} - \frac{55}{3e^2} + \frac{35}{2e^4}\right)(1-e^2),$$

$$\omega_{10} = \frac{\sqrt{1-e^2}}{e}\left(16 - \frac{27}{2e^2} + \frac{10}{e^4}\right)\arcsin e - \left(\frac{1}{3} - \frac{41}{6e^2} + \frac{10}{e^4}\right)(1-e^2).$$

The graphs of dependence of each of the corrections for the angular velocity on the eccentricity is presented in Fig. 9.

Thus, in the space of constant (positive) curvature the homogeneous liquid self-gravitating spheroid cannot rotate as a rigid body, and the angular velocity distribution of fluid particles depends only on the distance to the symmetry axis: $\omega = \omega(r)$.

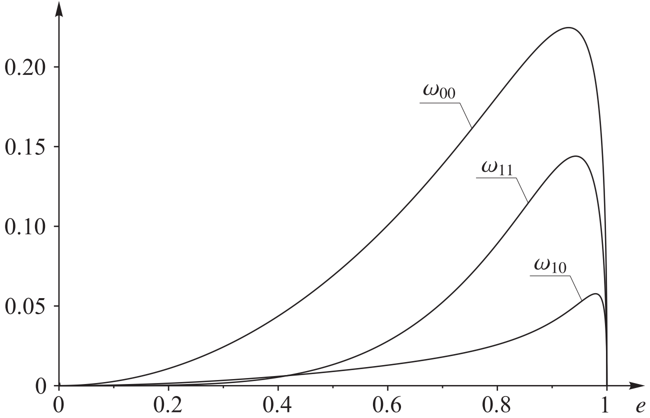


Рис. 9: Dependences of ω_{00}, ω_{11}, and ω_{10} on the eccentricity e.

Remark

For completeness we also present the equations which describe axisymmetric figures of equilibrium in curvilinear orthogonal coordinates (μ, ν, φ) *and are defined as follows:*

$$\frac{x_0^2}{d^2} = \frac{(\delta-\mu)(\delta+\nu)}{\delta+1}, \quad \frac{x_1^2}{d^2} = \mu\nu, \quad \delta = \frac{R^2}{d^2}$$

$$\frac{x_2^2}{d^2} = \frac{(1+\mu)(1-\nu)}{\delta+1}\cos^2\varphi, \quad \frac{x_3^2}{d^2} = \frac{(1+\mu)(1-\nu)}{\delta+1}\sin^2\varphi, \quad 0<\mu<\delta, \quad 0<\nu<1.$$

In this case the system (80) takes the form

$$\frac{\partial U}{\partial \mu} + \frac{1}{\rho(\mu)}\frac{\partial p}{\partial \mu} = -\frac{\delta d^2}{2(\delta+1)}(1-\nu)\omega^2, \quad \frac{\partial U}{\partial \nu} + \frac{1}{\rho(\mu)}\frac{\partial p}{\partial \nu} = \frac{\delta d^2}{2(\delta+1)}(1+\mu)\omega^2,$$

$$\Delta_{\mu\nu} U(\mu,\nu) = 4\pi G\rho(\mu),$$

$$R^2\Delta_{\mu\nu} = \frac{4}{\mu+\nu}\left(\sqrt{\mu(\delta-\mu)}\frac{\partial}{\partial\mu}\left((1+\mu)\sqrt{\mu(\delta-\mu)}\frac{\partial}{\partial\mu}\right) + \right.$$
$$\left. + \sqrt{\nu(\delta+\nu)}\frac{\partial}{\partial\nu}\left((1-\nu)\sqrt{\nu(\delta+\nu)}\frac{\partial}{\partial\nu}\right)\right).$$

This form of equations is preferable if it is necessary to obtain a solution in terms of quadratures (and not in the form of a power series).