Dynamics of self-gravitating ellipsoids

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Abstract. We investigate the figures of equilibrium of a self-gravitating ideal fluid with a stratified density and a steady-state velocity field. As in the classical formulation of the problem, it is assumed that the figures, or their layers, uniformly rotate about an axis fixed in space.

Introduction

This paper is concerned with exact solutions to the problem of (axisymmetric) figures of equilibrium of a self-gravitating ideal fluid with density *stratification*. First of all, we briefly recall the well-known results:

For *homogeneous* fluid, the following ellipsoidal equilibrium figures for which the entire mass *uniformly rotates as a rigid body* about a fixed axis are well known: the Maclaurin spheroid (1742), the Jacobi ellipsoid (1834). In addition, in the case of a homogeneous fluid there also exist *figures of equilibrium with internal flows*: the Dedekind ellipsoid (1861), the Riemann ellipsoids (1861).

On the other hand, Hamy [3], Volterra [4] and Pizzetti [5] showed that for a stratified fluid mass rotating as a rigid body there exist no figures of equilibrium in the class of ellipsoids.

Hamy proved this theorem for the case of a finite number of ellipsoidal layers with constant density, Volterra generalized this result to the case of continuous density distribution for a homothetic stratification of ellipsoids, and Pizzetti gave the simplest and most rigorous proof in the general case for both continuous and piecewise constant density distribution.

Inhomogeneous figures with isodensity distribution of the angular velocity of layers

If one admits the possibility that the angular velocity of fluid particles is not constant for the entire fluid mass, then equilibrium figures for an arbitrary axisymmetric form of the surface and density stratification are possible. For example, Chaplygin [2] explicitly showed a spheroidal equilibrium figure with a nonuniform distribution of angular velocities for the case of homothetic density stratification. It turns out that the surfaces with equal density $\rho(\vec{r}) = \text{const.}$ do not coincide with the surfaces of equal angular velocity $\omega(\vec{r}) = \text{const.}$ S. A. Chaplygin tried to use the resulting solution to explain the dependence of the angular velocity of rotation of the outer layers of the Sun on the latitude.

In [6] an explicit solution of another kind was found for which the equilibrium figure is a spheroid consisting of two fluid masses of different density $\rho_1 \neq \rho_2$ separated by the spheroidal boundary confocal to the outer surface, with each layer rotating at constant angular velocity such that $\omega_1 \neq \omega_2$. A generalization of this solution to the case of an arbitrary finite number of "confocal layers" was obtained by Esteban [1].

In this paper we obtain a generalization of this solution to the case of an arbitrary confocal (both continuous and piecewise constant) density stratification. For comparison, we also present Chaplygin's solution for the homothetic stratification.

For an arbitrary confocal stratification the angular velocity on the outer surface of the inhomogeneous spheroid is the same as the angular velocity ω_0 of the Maclaurin spheroid with density $\langle\rho\rangle$:

$$\frac{\omega_0^2}{2\pi G\langle\rho\rangle} = \mu_0((1+3\mu_0^2)\text{arcctg}(\mu_0) - 3\mu_0), \quad \mu_0 = \frac{b}{\sqrt{a^2-b^2}} \tag{1}$$

where $\langle\rho\rangle$ is the average density of the spheroid.

To keep track of the dependence of the angular velocity of the layers on the change in density, we consider an inhomogeneous spheroid with different functions of density distribution of the following form:

$$\rho(\mu) = \rho_n^{(0)}(1-\alpha_n\mu^n), \ n = 2, 4, 6, \tag{2}$$

where $\rho_n^{(0)}$ and α_n are some constants ($\rho_n^{(0)}$ has the meaning of density at the center of the spheroid). We will determine their values from the given average density of the body $\langle\rho\rangle = \frac{\int \rho dV}{\int dV}$ and the given ratio between the density on the surface and the average density of the body $\varepsilon = \frac{\langle\rho\rangle}{\rho(\mu_0)}$,

$$\alpha = \frac{(1+n)(3+n)(1+\mu_0^2)(1-\varepsilon)\mu_0^{-n}}{(3+n)(1-\varepsilon(1+n)(1+\mu_0^2)) + 3(1+n)\mu_0^2}$$

$$\rho_0 = \langle\rho\rangle\frac{(3+n)(\varepsilon(1+n)(1+\mu_0^2)-1) - 3(1+n)\mu_0^2}{n\varepsilon((1+n)\mu_0^2+3+n)}.$$

As an example, assume that the eccentricity e_0 and the quantity ε, which are the same as the data of the Earth:

$$e_0 = 0.08181, \quad \varepsilon = 2.5.$$

Figure 1 shows the dependences of $\frac{\rho}{\langle\rho\rangle}$ on the coordinate of the layer μ for (2). As we can see, the density increases most sharply at the center of the spheroid for $n = 2$ and then, as n increases, the density decreases.

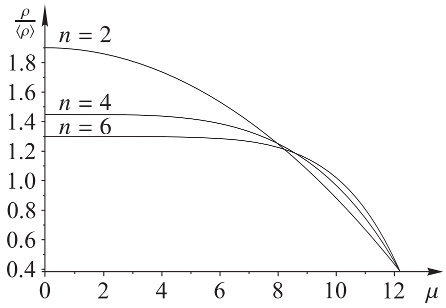

FIGURE 1. A graph showing the dependence of the relation $\frac{\rho}{\langle\rho\rangle}$ on the layer μ

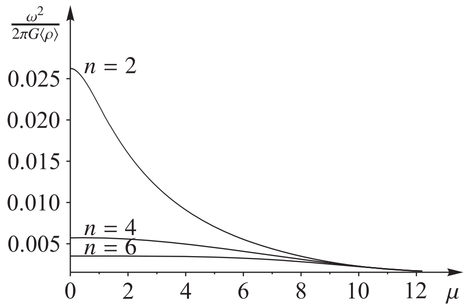

FIGURE 2. A graph showing the dependence of the angular velocity on the layer μ

To find the angular velocity, we substitute the density distributions and obtain the dependence of the angular velocity on the layer. A graph of this dependence is shown in Fig. 2. Since the explicit formulae for $\omega(\mu)$ are unwieldy, we do not present them here.

For the angular velocity with density distribution (2) one may draw the following conclusion from Fig. 2: *the closer the center of the spheroid, the larger the angular velocity; specifically, the larger the value of density at the center of the spheroid (with $n = 2$), the larger the increase in the angular velocity.*

References

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