Bifurcation diagram and a qualitative analysis of particle motion in a Kerr metric

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A Kerr metric is a stationary and cylindrically symmetric solution of the Einstein equations in vacuum. In the Boyer – Lindquist coordinates $\boldsymbol{x} = (t, r, \theta, \varphi)$ the Kerr metric is represented in the following form

$$g_{ij} = \begin{bmatrix} \dfrac{\Delta(r) - a^2 \sin^2\theta}{\rho^2} & 0 & 0 & \dfrac{2ar}{\rho^2}\sin^2\theta \\ 0 & -\dfrac{\rho^2}{\Delta(r)} & 0 & 0 \\ 0 & 0 & -\rho & 0 \\ \dfrac{2ar}{\rho^2}\sin^2\theta & 0 & 0 & \dfrac{\sin^2\theta}{\rho^2}\left(\Delta a^2 \sin^2\theta - (a^2 + r^2)^2\right) \end{bmatrix},$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta(r) = r^2 - 2r + a^2$. In the Kerr metric the coordinates r and t are measured in the following units: $\frac{Gm}{c^2}$, $\frac{Gm}{c^3}$, where m is the mass of the celestial body. The variables $\theta \in (0, \pi)$, $\varphi \in [0, 2\pi)$ are angle variables. The dimensionless parameter a is expressed in terms of the angular momentum of the celestial body M_z relative to the symmetry axis as follows: $a = \frac{cM_z}{Gm^2}$. If $a = 0$ (i.e., if there is no rotation), the metric Kerr becomes a Schwarzschild metric.

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[2] Carter B. Global structure of the Kerr family of gravitational fields, Physical Review, 1968, vol. 174, no. 5, p. 1559.

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[4] Новиков И. Д., Фролов В. П., Физика черных дыр, Москва, 1986.

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For large distances, i.e., for $r \gg 1$ the metric becomes the flat Minkowski metric:

$$ds^2 = -dt^2 + \rho^2 \left(\frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta d\varphi^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (1)$$

where the Cartesian coordinates are given by the following relations:

$$x = \sqrt{r^2 + a^2} \sin\theta \cos\varphi, \quad y = \sqrt{r^2 + a^2} \sin\theta \sin\varphi, \quad z = r\cos\theta. \quad (2)$$

As can be seen, for (1) the levels surfaces of the *radial coordinate* $r = \mathrm{const}$ with $t = \mathrm{const}$ are confocal spheroids in the three-dimensional Euclidean space

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \quad (3)$$

Motion problem particles in the Kerr metric in the Newtonian limit (i.e., if $|\boldsymbol{v}| \ll c$, where $\boldsymbol{v}$ is the velocity of a particle) reduces to that of a point moving in the potential field

$$U = -\frac{G}{2} \left(\frac{m}{\sqrt{x^2 + y^2 + (z + ia)^2}} + \frac{m}{\sqrt{x^2 + y^2 + (z - ia)^2}} \right),$$

two fixed gravitational centers "located at imaginary points"(this is a special case of the Euler problem).

[8] Алексеев В.М. Обощенная пространственная задача двух неподвижных центров. Классификация движений, Бюллетень Ин-та теор. астрономии. 1965. Т. 10, № 4 (117). с. 241-271.

As is known, the Kerr metric has *event horizon*

$$\mathcal{S}_h = \{(t, r, \theta, \varphi) \mid r = r_+\},\ r_+ = 1 + \sqrt{1 - a^2}, \tag{4}$$

where r_+ is the largest root of the equation $\Delta(r) = 0$.
From a geometrical point of view, the horizon $\mathcal{S}_h$ determines an isotropic hypersurface (null hypersurface) in spacetime whose section $\mathcal{S}_h^t$ is, for an external observer, diffeomorphic to the two-dimensional sphere $\mathbb{S}^2$ at any instant of time $t = \mathrm{const}$ (since the Kerr metric in the Boyer – Lindquist coordinates is stationary in these variables, the sphere $\mathcal{S}_h^t$ does not depend on t). In addition, for each point at the event horizon $\mathcal{S}_h$ the light cone (from the region of the future) lies entirely in the region $r \leqslant r_+$, and hence the world lines both of particles and of light beams will, after reaching $r = r_+$, no longer be able to return to the region $r > r_+$.
Since the region $r \leqslant r_+$ turns out to be inaccessible to the external observer, we will throughout consider the motion of particles only outside the event horizon of the Kerr metric.

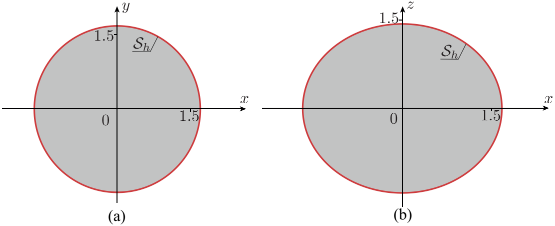


Рис. 1: A typical view of sections of the event horizon $\mathcal{S}_h$: (a) sections formed by the intersection with the plane $z = 0$, (b) sections formed by the intersection with the plane $y = 0$.

We use a natural parameterization of the trajectory of particle $\boldsymbol{x}(\tau)$, where τ is the proper time of the particle. Then the required trajectories satisfy equations:

$$\frac{d\boldsymbol{x}}{d\tau} = \frac{\partial H}{\partial \boldsymbol{p}}, \quad \frac{d\boldsymbol{p}}{d\tau} = -\frac{\partial H}{\partial \boldsymbol{x}}, \; H = \frac{1}{2} g^{ij} p_i p_j, \tag{5}$$

where $\boldsymbol{p} = (p_t, p_r, p_\theta, p_\varphi)$ is the momentum of the particle and g^{ij} is the matrix inverse to the metric.

The metric g_{ij} does not depend explicitly on time t and angle φ, therefore, they are cyclic coordinates for equations (5). As a consequence, the corresponding momenta remain unchanged:

$$E = -p_t = \text{const}, \quad L = p_\varphi = \text{const}.$$

From a physical point of view E is the energy of the material point and L is the projection of its angular momentum onto the symmetry axis.

Thus, the Hamiltonian system with two degrees of freedom decouples from the system

$$\frac{dp_r}{d\tau} = -\frac{\partial H}{\partial r}, \quad \frac{dp_\theta}{d\tau} = -\frac{\partial H}{\partial \theta}, \quad \frac{dr}{d\tau} = \frac{\partial H}{\partial p_r}, \quad \frac{d\theta}{d\tau} = \frac{\partial H}{\partial p_\theta},$$

$$H = \frac{1}{2\rho^2}\left(\frac{p_r^2}{\Delta(r)} + p_\theta^2\right) + V, \tag{6}$$

$$V = \frac{1}{2\Delta(r)\rho^2}\left(-\big(\Delta(r)\rho^2 + 2r(r^2+a^2)\big)E^2 + 4arEL + \left(\frac{\Delta(r)}{\sin^2\theta} - a^2\right)L^2\right).$$

and the phase space of this system has the form

$$\mathcal{M}^4 = \{\boldsymbol{z} = (p_r, p_\theta, r, \theta) \mid r \in (r_+, +\infty), \theta \in (0, \pi)\}.$$

The trajectories of the material particles lie on the fixed level set of the Hamiltonian

$$H = -\frac{1}{2}.$$

In addition to the Hamiltonian H, the reduced system (6) has the additional Carter integral:

$$F(\boldsymbol{z}) = p_\theta^2 + \left(aE\sin\theta - \frac{L}{\sin\theta}\right)^2 - 2a^2 H(\boldsymbol{z})\cos^2\theta. \tag{7}$$

Since both integrals of the system (6) turn out to be quadratic, it can, as is well known, be integrated by the method of separation of variables. In this case, r and θ are separating variables, and therefore, to reduce the problem to quadratures, we fix the common level set of the Carter integral and the value of the Hamiltonian:

$$H(\boldsymbol{z}) = -\frac{1}{2}, \quad F(\boldsymbol{z}) = Q + (L - aE)^2, \tag{8}$$

where Q is some constant.

Now we express, taking into account (8) from (7), the momenta p_r and p_θ and substitute them into the last two equations of motion of the system (6). As a result, we obtain equations of motion for r and θ in the following form:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{\rho^4}R(r), \quad \left(\frac{d\theta}{d\tau}\right)^2 = \frac{1}{\rho^4}\Theta(\theta),$$
$$R(r) = \big(E(r^2+a^2) - aL\big)^2 - (Q + (L-aE)^2 + r^2)\Delta(r), \tag{9}$$
$$\Theta(\theta) = Q - \cos^2\theta\left(a^2(1-E^2) + \frac{L^2}{\sin^2\theta}\right).$$

The trajectories of the reduced system lie on two-dimensional integral submanifolds

$$\mathcal{M}^2_{\boldsymbol{I}} = \left\{(p_r, p_\theta, r, \theta) \mid H(\boldsymbol{z}) = -\frac{1}{2},\ F(\boldsymbol{z}) = Q + (L-aE)^2\right\}, \tag{10}$$

where $\boldsymbol{I} = (L, E, Q)$. Integral submanifold is represented as the product of a pair of plane curves:

$$\mathcal{M}^2_{\boldsymbol{I}} = \mathcal{C}^1_r \times \mathcal{C}^1_\theta,$$
$$\mathcal{C}^1_r = \{(p_r, r) \mid p_r^2 = \Delta^2(r)R(r),\ r > r_+\}, \quad \mathcal{C}^1_\theta = \{(p_\theta, \theta) \mid p_\theta^2 = \Theta(\theta),\ 0 < \theta < \pi\}, \tag{11}$$

which depend on four parameters: a, L, E and Q. Consequently, the problem of restructuring $\mathcal{M}^2_{\boldsymbol{I}}$ reduces to separately investigating the bifurcations of each of the curves, $\mathcal{C}^1_r$ and $\mathcal{C}^1_\theta$.

From the known solutions $r(\tau)$ and $\theta(\tau)$ the evolution of the other variables is defined, according to (5), using the quadratures

$$
\begin{aligned}
\rho^2 \frac{d\varphi}{d\tau} &= \frac{a}{\Delta(r)} \big(E(r^2 + a^2) - aL\big) - aE + \frac{L}{\sin^2\theta}, \\
\rho^2 \frac{dt}{d\tau} &= \frac{r^2 + a^2}{\Delta(r)} \big(E(r^2 + a^2) - aL\big) + aL - a^2 E \sin^2\theta.
\end{aligned}
\tag{12}
$$

Analysis of the curve $\mathcal{C}_\theta^1$

We first note that the equation for $\mathcal{C}_\theta^1$ can be represented as

$$p_\theta^2 + U_\theta = Q = \text{const}, \quad U_\theta = Q - \Theta(\theta) = \cos^2\theta \left[a^2(1 - E^2) + \frac{L^2}{\sin^2\theta} \right].$$

Since U_θ does not depend on Q, the analysis of the curves $\mathcal{C}_\theta^1$ is entirely similar to analysis of trajectories in the phase space $\mathbb{R}^2 = \{(\theta, p_\theta)\}$ for the Hamiltonian system describing the motion of a material point in a potential field U_θ (the function U_θ is sometimes called a *latitudinal potential*), and the constant Q is similar to the level set of the energy integral. As is well known, ascertaining the type of these trajectories reduces to analyzing the behavior of the function U_θ on the interval $\theta \in (0, \pi)$.
Firstly, we note that the function U_θ is symmetric about the straight line $\theta = {}^{\pi}/_{2}$, i.e.,

$$U_\theta\left(\frac{\pi}{2} + x\right) = U_\theta\left(\frac{\pi}{2} - x\right).$$

Secondly, if $\theta = {}^{\pi}/_{2}$, this function vanishes and simultaneously has a critical point:

$$U_\theta\left(\frac{\pi}{2}\right) = 0, \quad \left.\frac{dU_\theta}{d\theta}\right|_{\theta={}^{\pi}/_{2}} = 0.$$

Thirdly, the numerator of the function is a fourth-order polynomial in the variable $u = \cos\theta$. Therefore, in addition to the (multiple) roots, the function $U_\theta(u)$ can have a pair of roots when $\theta = {}^{\pi}/_{2}$ $(u = 0)$.

Let us define the constant value

$$C_1 = a^2(E^2 - 1) - L^2,$$

which is proportional to the second derivative of the function U_θ in $\theta = {}^{\pi}/_{2}$. Depending on the sign of C_1 and the value of L, four qualitatively different types of the function U_θ are possible, each of which corresponds to the family of curves, which are parameterized by the value of the integral Q.

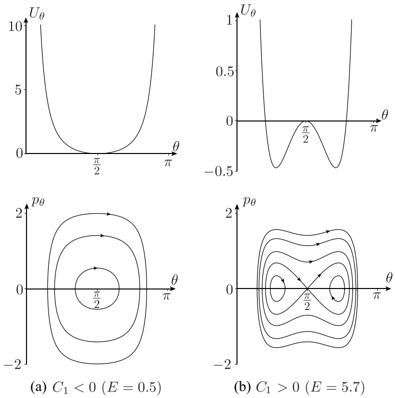


Рис. 2: A typical view of the latitudinal potential U_θ and the corresponding curves $\mathcal{C}^1_\theta$ on the plane $(\theta,\, p_\theta)$ with $L \neq 0$ and different values of C_1 for the fixed $a = 0.3$ and $L = 1$.

Since the function $\Delta(r)$ is always positive on the interval $r \in (r_+, +\infty)$, the analysis of the curve $\mathcal{C}_r^1$ reduces in fact to investigating the behavior of the zeroes of the function

$$\begin{aligned} R(r) = \big(E(r^2+a^2)-aL\big)^2 - (Q+(L-aE)^2+r^2)\Delta(r) = \\ = (E^2-1)r^4 + 2r^3 + \big(a^2(E^2-1)-L^2-Q\big)r^2 + 2\big(Q+(L-aE)^2\big)r - a^2Q \end{aligned} \tag{13}$$

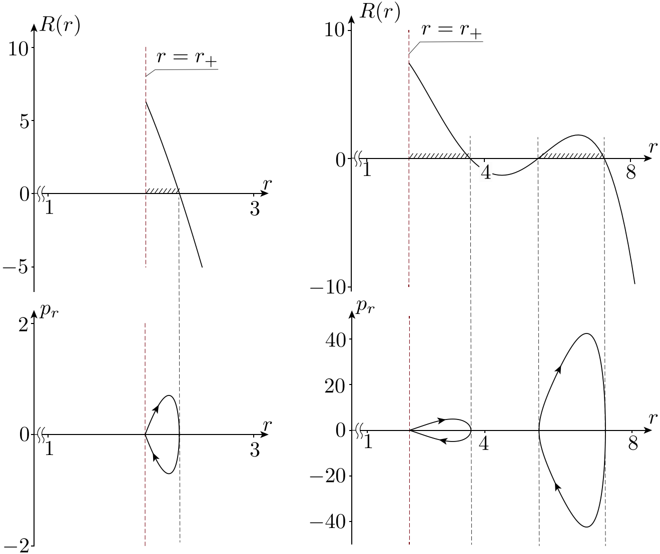
depending on the parameters a, Q, E and L.

This function is a fourth-degree polynomial and therefore has no more than 4 roots. We also note that

$$R(r_+) \geqslant 0, \tag{14}$$

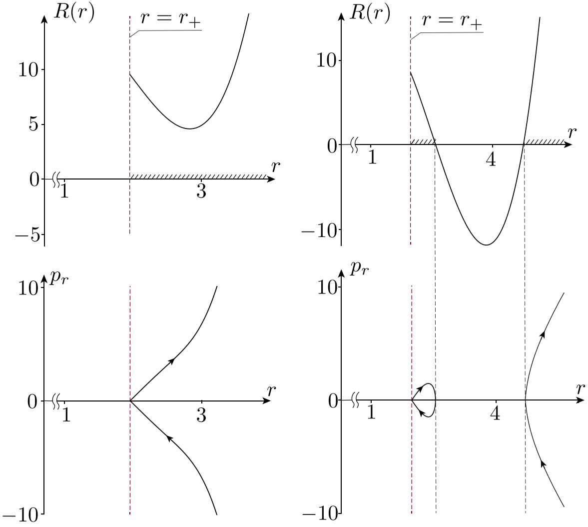
on the interval $r \in (r_+, +\infty)$ the function $R(r)$ can have:

- if $E^2 < 1$, either 1 or 3 roots,
- if $E^2 > 1$, either 0 or 2 roots.



(a) $L = 4, E = 0.95, Q = 1$ (b) $L = 3.2, E = 0.937, Q = 0.9$

Рис. 3: The function $R(r)$ and the corresponding curves $\mathcal{C}_r^1$ on the plane $(r,\, p_r)$ for the fixed $a = 0.3$ with $E < 1$.



(a) $E = 1.1$ (b) $E = 1.03$

Рис. 4: The function $R(r)$ and the corresponding curves $\mathcal{C}_r^1$ on the plane $(r,\ p_r)$ for the fixed $a = 0.3$, $L = 4$, $Q = 1$ with $E > 1$.

$r = r_c = \mathrm{const}$ which are critical points $R(r)$

$$R(r_c) = 0, \quad \left.\frac{dR}{dr}\right|_{r=r_c} = 0, \quad r_c > r_+. \tag{15}$$

This yields a surface in the space of first integrals $\mathbb{R}^3_I$ which consists of two parts

$$\begin{gathered}\Sigma_0^r = \Sigma_+^r \cup \Sigma_-^r, \\ \Sigma_+^r = \{(L, E, Q) \mid E = E_+(r_c, Q),\ L = L_+(r_c, Q),\ Q \geqslant 0\}, \\ \Sigma_-^r = \{(L, E, Q) \mid E = E_-(r_c, Q),\ L = L_-(r_c, Q),\ Q \geqslant 0\}.\end{gathered}$$

Remark 1

If $Q = 0$, then after simplifications the equations for Σ_0^r can be represented as

$$\begin{aligned} E_\pm &= \frac{r_c^{3/2} - 2r_c^{1/2} \pm a}{r_c^{3/4}\left(r_c^{3/2} - 3r_c^{1/2} \pm 2a\right)^{1/2}}, \\ L_\pm &= \pm\frac{r_c^2 \mp 2ar_c^{1/2} + a^2}{r_c^{3/4}\left(r_c^{3/2} - 3r_c^{1/2} \pm 2a\right)^{1/2}}, \end{aligned} \tag{16}$$

where the upper sign refers to E_+ and L_+, and the lower sign, to E_- and L_-.

Trajectories in the equatorial plane $Q = 0$

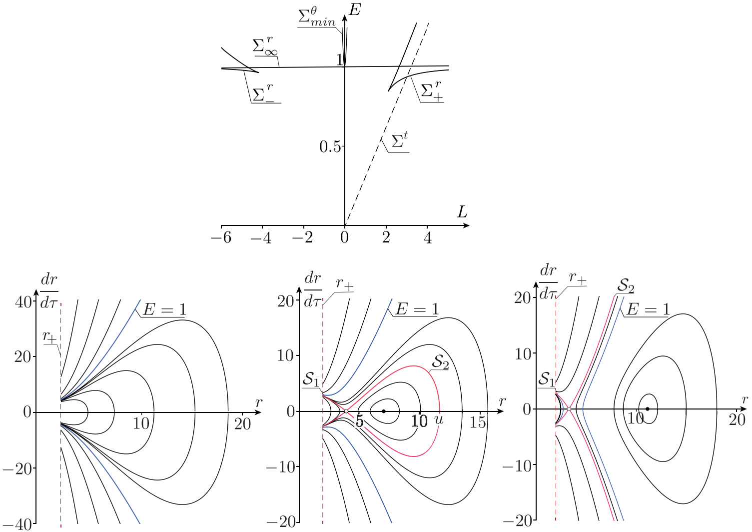


Рис. 5: Three possible types of phase portraits of the reduced system for a fixed value of L ($a = 0.3$).

Assume that the values of the integrals L and E lie in a region bounded by the curves $\Sigma^r_\pm$ and $E < 1$, three roots of the polynomial $R(r)$ lie on the interval $(r_+, +\infty)$ in this case:

$$R(r) = (E^2 - 1)(r - r_u^{(1)})(r - r_u^{(2)})(r - r_u^{(3)})r, \ r_+ < r_u^{(1)} < r_u^{(2)} < r_u^{(3)}.$$

and the segment $[r_u^{(2)}, r_u^{(3)}]$ defines bounded trajectories. We transform to the angle variable $\psi \in (0, 2\pi]$:

$$r = \frac{r_u^{(3)} - r_u^{(2)}}{2} \cos\psi + \frac{r_u^{(3)} + r_u^{(2)}}{2}.$$

We finally obtain equations governing the motion on invariant tori $\mathbb{T}^2 = \{(\psi, \varphi) \mod 2\pi\}$ in phase space in the following form:

$$\frac{d\psi}{du} = d\sqrt{(\Gamma_1 + \cos\psi)(\Gamma_2 + \cos\psi)}, \ \frac{d\varphi}{du} = \Phi(\psi) = r\frac{2aE + L(r-2)}{\Delta(r)}\bigg|_{r=r(\psi)},$$
$$d = \frac{\sqrt{1-E^2}}{2}(r_u^{(3)} - r_u^{(2)}), \ \Gamma_1 = \frac{r_u^{(2)} + r_u^{(3)}}{r_u^{(3)} - r_u^{(2)}} > 1, \ \Gamma_2 = \frac{r_u^{(2)} + r_u^{(3)} - 2r_u^{(1)}}{r_u^{(3)} - r_u^{(2)}} > 1. \tag{17}$$

It is the rotation number that allows one to classify the trajectories on $\mathbb{T}^2$ depending on parameters. In this case the rotation number can be represented as

$$\rho_{L,E} = 2\pi d\left[\int\limits_0^{2\pi} \frac{\Phi(\psi)d\psi}{\sqrt{(\Gamma_1 + \cos\psi)(\Gamma_2 + \cos\psi)}}\right]^{-1}.$$

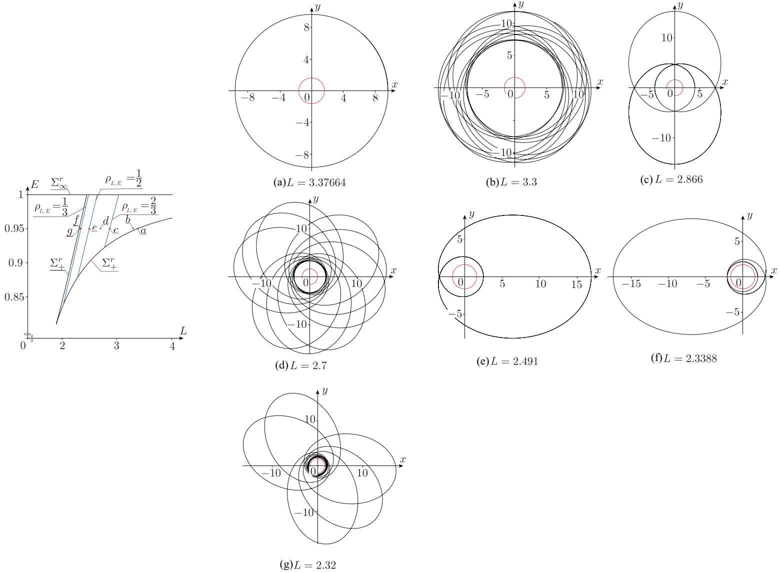


Рис. 6: Curves for the fixed $a = 0.95$ on the plane L, E which correspond to the rational values of the rotation number $\rho_{\varphi/r}$, and the trajectories in the equatorial plane for the fixed $E = 0.95$ and different L.

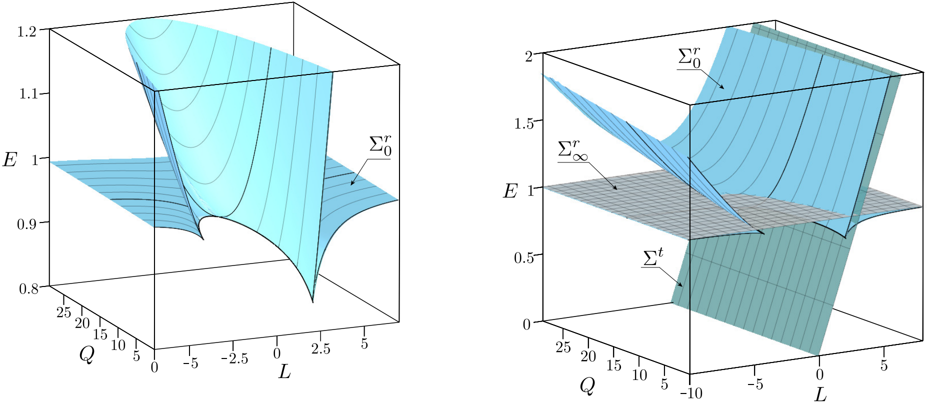


Рис. 7: Arrangement of bifurcation surfaces and curves in the space of first integrals $\mathbb{R}^3_{\boldsymbol{I}}$ for the fixed $a = 0.9$. Gray denotes the region of possible values of L, E and Q.

The trajectories of of light rays lie on the fixed level set of the Hamiltonian $H = 0$.

$$\begin{gathered}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{E^2\rho^4}\left[r^4 + (a^2 - \lambda^2 - q)r^2 + 2((a-\lambda)^2 + q)r - a^2 q\right], \\ \left(\frac{d\theta}{d\tau}\right)^2 = \frac{1}{E^2\rho^4}\left[a^2\cos^2\theta - \lambda^2\cot^2\theta + q\right], \\ \lambda = \frac{L}{E}, \quad q = \frac{Q}{E^2}\end{gathered} \tag{18}$$

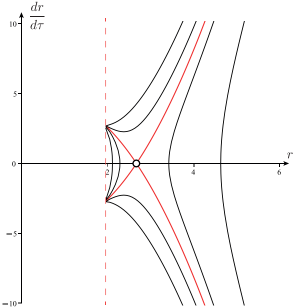
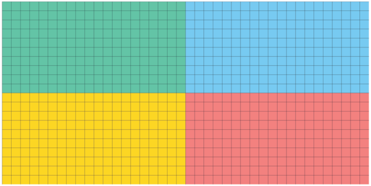
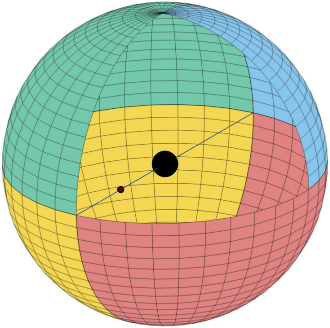
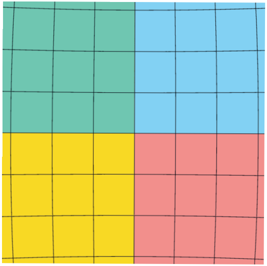


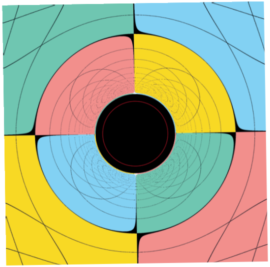
Рис. 8: Trajectories of the system versus q for fixed $\lambda = 4$, $a = 0.3$.

Ray tracing in case $a = 0$ (Schwarzschild)









Thank you for your attention!