Bifurcation diagram of particle motion in a Kerr metric

Ivan Bizyaev and Ivan Mamaev

Abstract. We investigate the dynamics of particles in a Kerr metric which describes the gravitational field in a neighborhood of a rotating black hole. After elimination of cyclic coordinates this problem reduces to investigating a Hamiltonian system with 2 degrees of freedom. This system possesses an additional Carter integral quadratic in momenta and hence is integrable by the Liouville – Arnold theorem. A bifurcation diagram is constructed and a classification of the types of trajectories of the system is carried out according to the values of first integrals.

Introduction

Integrability of the geodesic flow in a Kerr metric was established by Carter [4] in 1968, and a large number of results have been obtained since then in this problem, see, e.g., the reviews [7]. However, a complete bifurcation diagram has been constructed recently in [2]. Using this diagram, an analysis of bifurcations of different types of the system's trajectories has been carried out for the case where its parameter values are varied. In addition, a graphical representation of possible types of motion depending on the values of the first integrals has been obtained. In what follows, our analysis of the trajectories of a material point will be based on [2].

At the same time, there are a number of particular results in this direction. For example, bifurcation curves for plane orbits have been obtained for the critical value of the Carter integral $Q = 0$ in [1] (in particular, r_{ISCO} (Innermost Stable Circular Orbit) was found for the Kerr metric), and a corresponding diagram was constructed in [8].

There are many other papers describing various special properties of (time-like) geodesics of the Kerr metric. We mention some of them which are related to our analysis. For example, in [3] the motion of particles falling from the state of rest was examined, and the author of [5] found numerically trajectories making a large number of turns in the neighborhood of a black hole and then receding from it (see also [6], where such orbits are called "zoom-whirl" orbits).

1. The Kerr metric

In the Boyer – Lindquist coordinates $\boldsymbol{x} = (t, r, \theta, \varphi)$ the Kerr metric is represented in the following form:

$$ds^2 = \frac{\Delta(r)}{\rho^2}\big(dt - a\sin^2\theta d\varphi\big)^2 - \frac{\sin^2\theta}{\rho^2}\big((r^2+a^2)d\varphi - a dt\big)^2 - \rho^2\left(\frac{dr^2}{\Delta(r)} + d\theta^2\right),$$
$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta(r) = r^2 - 2r + a^2, \tag{1}$$

where $\alpha, \beta = 0, 1, 2, 3$, summation is implied over repeated indices, and the signature $(1, 3)$ has been chosen.

The dimensionless parameter a is expressed in terms of the angular momentum of the celestial body M_z relative to the symmetry axis as follows:

$$a = \frac{cM_z}{Gm^2}.$$

If $a = 0$ (i.e., if there is no rotation), the metric (1) becomes a Schwarzschild metric.

As a result, we obtain equations of motion for r and θ in the following form:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{\rho^4}R(r), \quad \left(\frac{d\theta}{d\tau}\right)^2 = \frac{1}{\rho^4}\Theta(\theta),$$
$$R(r) = \big(E(r^2+a^2) - aL\big)^2 - (Q + (L - aE)^2 + r^2)\Delta(r), \tag{2}$$
$$\Theta(\theta) = Q - \cos^2\theta\left(a^2(1-E^2) + \frac{L^2}{\sin^2\theta}\right).$$

From a physical point of view, E is the energy of the material point, and L is the projection of its angular momentum onto the symmetry axis of the metric, Q is constant Carter integral.

As can be seen, in order to integrate these equations in explicit form, one needs to rescale time as $d\tau = \rho^2(r,\theta)du$.

From the known solutions $r(\tau)$ and $\theta(\tau)$ the evolution of the other variables is defined using the quadratures

$$\rho^2\frac{d\varphi}{d\tau} = \frac{a}{\Delta(r)}\big(E(r^2+a^2) - aL\big) - aE + \frac{L}{\sin^2\theta},$$
$$\rho^2\frac{dt}{d\tau} = \frac{r^2+a^2}{\Delta(r)}\big(E(r^2+a^2) - aL\big) + aL - a^2E\sin^2\theta. \tag{3}$$

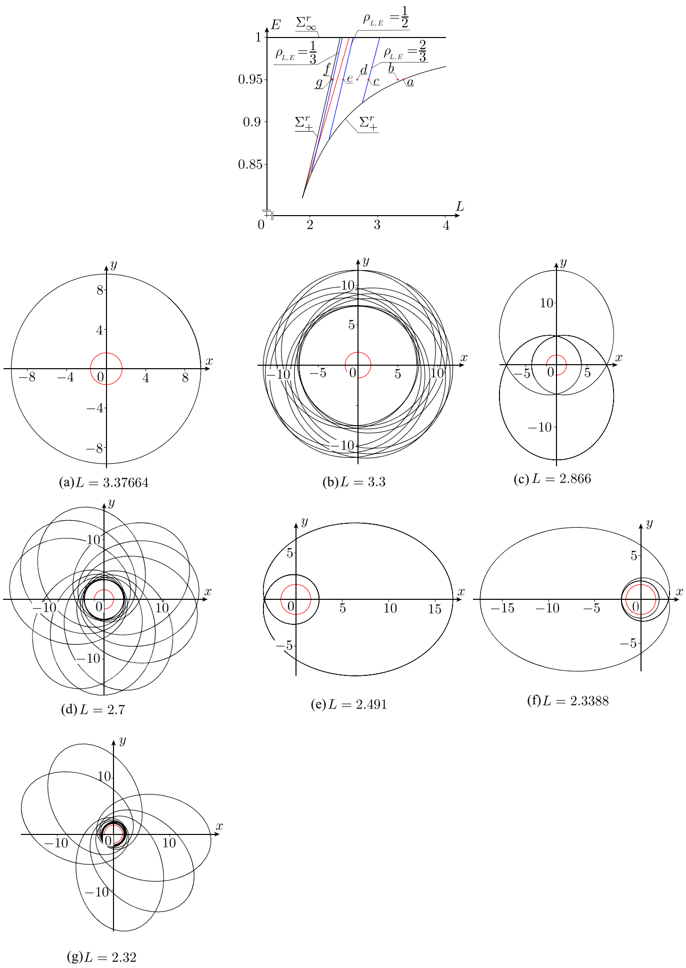

FIGURE 1. Curves for the fixed $a = 0.95$ on the plane L, E which correspond to the rational values of the rotation number $\rho_{\varphi/r}$, and the trajectories in the equatorial plane for the fixed $E = 0.95$ and different L.

2. Trajectories in the equatorial plane

Let the value of the Carter integral be zero, $Q = 0$. Then it follows from the analysis of of the latitudinal motion that there exist trajectories lying in the equatorial plane $\theta = \frac{\pi}{2}$, and that all of them are critical (since in this case the latitudinal potential has a critical point).

The system of equations, which governs the evolution of the angles ψ and φ, defines a vector field on the torus $\mathbb{T}^2$ without fixed points. It is the rotation number that allows one to classify the trajectories on $\mathbb{T}^2$ depending on parameters. In this case the rotation number can be represented as

$$\rho_{L,E} = 2\pi d \left[\int\limits_0^{2\pi} \frac{\Phi(\psi) d\psi}{\sqrt{(\Gamma_1 + \cos\psi)(\Gamma_2 + \cos\psi)}} \right]^{-1}.$$

If $\rho_{L,E}$ takes a rational value, then all trajectories on the corresponding invariant torus $\mathbb{T}^2$ with given values of L and E are periodic. If $\rho_{L,E}$ takes an irrational value, then the trajectories on the torus $\mathbb{T}^2$ are quasi-periodic. The curves on the plane L, E which correspond to the rational values of the rotation number equal to $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$ are shown in Fig. 1.

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Ivan Bizyaev
Ural Mathematical Center
Udmurt State University
Universitetskaya 1, Izhevsk 426034, Russia
e-mail: bizyaevtheory@gmail.com

Ivan Mamaev
Ural Mathematical Center
Udmurt State University
Universitetskaya 1, Izhevsk 426034, Russia
e-mail: mamaev@rcd.ru