Chaotic behavior in the generalized n-center problem

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Abstract. We consider a Hamiltonian system with Hamiltonian $H = \|p\|^2/2 + V(q)$. The configuration space M is a 2-dimensional manifold (for noncompact M certain conditions at infinity are required). It was proved in [2] that if the potential energy V has $n > 2\chi(M)$ Newtonian singularities, then the system is not integrable and has positive topological entropy on energy levels $H = h > \sup V$. We generalize this result to the case when the potential energy has several singular points $\Delta = \{a_1, \dots, a_n\}$ of type $V(q) \sim -\mathrm{dist}(q, a_j)^{-\alpha_j}$. As an application, we consider the generalized n-center problem in $\mathbb{R}^2$ and discuss possible extensions to the spatial n-center problem.

Our research is motivated by the generalized n-center problem. Let

$$H(q,p) = \frac{1}{2}|p|^2 + V(q), \quad V(q) = -\sum_{j=1}^{n} \frac{m_j}{|q - a_j|^{\alpha_j}} + U(q), \quad q \in \mathbb{R}^2.$$

Then we have:

- $\alpha_j = 1$, $n = 2$, and $U = 0$ – integrable 2 center problem.
- $\alpha_j = 1$ (Newtonian singularities) and $n \geq 3$ – there exists chaotic invariant set on energy levels $H = h > \sup V$ [2, 3].
- $\alpha_j > 2$ (strong singularities) and $n \geq 2$ – chaotic invariant set for $h > \sup V$.

We consider a Hamiltonian system with 2-dimensional configuration space M and Hamiltonian $H = \|p\|^2/2 + V(q)$. The kinetic energy is given by a Riemannian metric (for noncompact M certain conditions at infinity are required). The potential energy V is a smooth function except at a finite number of singular points $\Delta = \{a_1, \dots, a_n\}$. Near a_j,

$$V(q) = -\frac{f_j(q)}{d(q, a_j)^{\alpha_j}} + U_j(q), \qquad f_j(a_j) > 0, \quad \alpha_j > 0.$$

Let $\chi(M)$ be the Euler characteristics of M. For Newtonian singularities we have the following old result.

Theorem 1. [2] *If $n > 2\chi(M)$, the system is non-integrable on energy levels $H = h > \sup V$.*

We may also add a 2-form of gyroscopic forces to the symplectic form $dp \wedge dq$. Our goal is to obtain similar non-integrability conditions for any $\alpha_j > 0$.

Polynomial in p and differentiable in q first integrals on an energy level $\{H = h\}$ are called Birkhoff conditional integrals. Let

$$S(\Delta) = \sum \alpha_j, \quad 1 \le \alpha_j < 2.$$

Theorem 2. [6] *Let M be a closed manifold and $h > \max V$.*

- *If $S(\Delta) > 2\chi(M)$, there are no nonconstant Birkhoff conditional integrals on the energy level $H = h$.*
- *If $S(\Delta) = 2\chi(M)$, such integrals may exist only when the gyroscopic form is exact.*

To prove chaotic behavior stronger conditions are needed.

Let $A_k = 2 - 2k^{-1}$, $k \in \mathbb{N}$, and let n_k be the number of singular points with $A_k \le \alpha_j < A_{k+1}$. Set $n_\infty = 2$. Denote

$$A(\Delta) = \sum_{2 \le k \le \infty} n_k A_k = n_2 + \frac{4}{3} n_3 + \frac{3}{2} n_4 + \frac{8}{5} n_5 + \cdots + 2n_\infty$$

We have $A(\Delta) \le S(\Delta)$ and $S(\Delta) = A(\Delta)$ iff all singularities are regularizable.

- If all singularities are weak with $0 < \alpha_j < 1$, then $A(\Delta) = 0$.
- If all singularities are Newtonian with $\alpha_j = 1$, then $A(\Delta) = n$.
- If all singularities are strong with $\alpha_j > 2$, then $A(\Delta) = 2n$.
- Newtonian singularities and Jacobi singularities ($\alpha_j = 2$) are critical.

For simplicity suppose there are no gyroscopic forces.

Theorem 3. [7] *If*

$$A(\Delta) > 2\chi(M),$$

then the system has a compact chaotic invariant set of noncollision trajectories on any energy level $H = h > \sup V$.

For noncompact M certain conditions at infinity are required.

This result is purely topological: almost no analytical properties of the potential, except the presence of singularities, are involved.

Corollary 1. *For the generalized n-center problem in $\mathbb{R}^2$, if $A(\Delta) > 2$, the system has a compact chaotic invariant set on any energy level $H = h > \sup V$.*

A weaker result was proved in [5]. For nonintegrability condition $S(\Delta) > 2$ is also sufficient. We do not know if this is enough for chaotic behavior.

Other examples:

- $M = \mathbb{T}^2$, $\chi(\mathbb{T}^2) = 0$. Theorem 3 works if there is a nonweak singularity with $\alpha \ge 1$. We do not know if the existence of a weak singularity on $\mathbb{T}^2$ always implies chaotic behavior.

- $M = S^2$, $\chi(S^2) = 2$. Theorem 3 works for:
  - $n \geq 5$ singularities with $\alpha_j \geq 1$,
  - $n \geq 4$ singularities with $\alpha_j \geq 4/3$,
  - $n \geq 3$ singularities with $\alpha_j \geq 3/2$.
  - 3 singularities with $\alpha_j \geq 1$ and the 4th with $\alpha_4 \geq 4/3$.

For $n = 4$ Newtonian singularities on S^2 the system may be integrable on an energy level $H = h > \max V$ [2].

The proof of Theorem 3 is based on on the generalized Levi-Civita regularization $q = a_j + z^\beta$, $z \in \mathbb{C}$.

Let

$$\Delta = \Delta_{weak} \cup \Delta_{newt} \cup \Delta_{mod} \cup \Delta_{jac} \cup \Delta_{strong}.$$

The most nontrivial are moderate singularities with $1 < \alpha_j < 2$. Trajectories on $\{H = h\}$ are geodesics of the Jacobi metric

$$g_h(q, \dot{q}) = \sqrt{2(h - V(q))}\|\dot{q}\|.$$

The Jacobi distance to the strong singularities is infinite, so they may be removed replacing M by $M \setminus \Delta_{strong}$.

Theorem 4. *There exists a surface $\hat{M}$, a K-sheet covering $\phi : \hat{M} \to M \setminus (\Delta_{jac} \cup \Delta_{strong})$ branched over the set $\Delta_{newt} \cup \Delta_{mod}$, and a smooth Riemannian metric on $\hat{M}$ such that:*

- *Projections to M of minimal geodesics on the universal covering of $\hat{M}$ are trajectories with energy $H = h$ having no collisions with Δ, except maybe with regularizable singularities Δ_{reg}.*
- *The Euler characteristics*

$$\chi(\hat{M}) = K\big(\chi(M) - \frac{1}{2}A(\Delta)\big) < 0.$$

Since $\chi(\hat{M}) < 0$, a modification of old results of Kozlov [1] may be applied to prove Theorem 3.

Our results can be partly extended to the spatial generalized n-center problem. For $n \geq 3$ Newtonian singularities in $\mathbb{R}^3$ the existence of a chaotic invariant set may be proved using global KS regularization [4]. It replaces the configuration space $M = \mathbb{R}^3$ by the 4-dimensional manifold

$$\hat{M} = (S^2 \times \mathbb{R}^2)\#(S^2 \times S^2)\# \ldots \#(S^2 \times S^2).$$

Then Gromov's theorem may be used to prove positive topological entropy. If there is a generalized n-center problem in $\mathbb{R}^3$ with $n \geq 3$ singularities of order $1 < \alpha_j < 2$, global KS regularization gives a system with configuration space $\hat{M}$ and weak singularities of order $0 < \tilde{\alpha}_j < \alpha_j$ [8]. Then we hope that a modification of Gromov's theorem can be applied to obtain a chaotic invariant set. The problem is that, contrary to the 2-dimensional case, we can't exclude that chaotic trajectories enter weak singularities. Nevertheless, we have:

Conjecture. Let

$$B_k = 2 - 2^{k-1}, \quad m_k = \#\{a_k : B_k \leq \alpha_j < B_{k+1}\}.$$

If

$$B(\Delta) = \sum_{1 \leq k \leq \infty} m_k B_k > 2,$$

then the generalized n center problem in $\mathbb{R}^3$ has positive topological entropy on energy levels $H = h > \sup V$.

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