Chaotic behavior in the generalized n-center problem

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Setup

Consider a Hamiltonian system with Hamiltonian

$$H(q,p) = \frac{1}{2}\|p\|^2 + V(q).$$

The configuration space is $M \setminus \Delta$, where M is an oriented surface. The potential energy has singularities $\Delta = \{a_1, \dots, a_n\}$. Near a_j,

$$V(q) = -\frac{f_j(q)}{d(q,a_j)^{\alpha_j}} + U_j(q), \qquad f_j(a_j) > 0, \quad \alpha_j > 0.$$

The symplectic form $dp \wedge dq + \Gamma$ may be twisted by a 2-form Γ of gyroscopic (magnetic) forces.

Plane n-center problem

$$H(q,p) = \frac{1}{2}|p|^2 + V(q), \quad V(q) = -\sum_{j=1}^{n} \frac{m_j}{|q-a_j|^{\alpha_j}} + U(q).$$

- $\alpha_j = 1$ and $U = 0$ – integrable 2 center problem.
- $\alpha_j = 1$ (Newtonian singularities) and $n \geq 3$ – chaotic invariant set on energy levels $H = h > \sup V$ (B 1984, Knauf 1987).
- $\alpha_j > 2$ (strong singularities) and $n \geq 2$ – chaotic invariant set for $h > \sup V$.
- Goal: similar conditions for any α_j.

Classification of singularities

- $0 < \alpha_j < 1$ – weak.
- $1 < \alpha_j < 2$ – moderate.
- $\alpha_j = 1$ – Newtonian.
- $\alpha_j = 2$ – Jacobian.
- $\alpha_j > 2$ – strong.

$$\Delta = \Delta_{weak} \cup \Delta_{newt} \cup \Delta_{mod} \cup \Delta_{jac} \cup \Delta_{strong}.$$

Newtonian and Jacobi singularities are critical.

Newtonian singularities

Let $\chi(M)$ be the Euler characteristics of M. For Newtonian singularities ($\alpha_j = 1$) we have:

Theorem (B 1984)

If $n > 2\chi(M)$, the system is non-integrable on energy levels $H = h > \sup V$.

Proof: Global Levi-Civita regularization replaces M by a surface $\hat{M}$ with $\chi(\hat{M}) < 0$. □

Birkhoff integrals

Polynomial in p and differentiable in q first integrals on an energy level are called Birkhoff conditional integrals. Let

$$S(\Delta) = \sum \alpha_j, \quad 1 \leq \alpha_j < 2.$$

For Newtonian singularities $S(\Delta) = n$.

Theorem (B-Kozlov 2017)

Let M be a closed manifold and $h > \max V$.

- *If $S(\Delta) > 2\chi(M)$, there are no nonconstant Birkhoff conditional integrals on $\{H = h\}$.*
- *If $S(\Delta) = 2\chi(M)$, such integrals may exist only when the gyroscopic form is exact: $\Gamma = d\omega$.*

For chaotic behavior stronger conditions are needed.

Conditions for positive topological entropy

$$A_k = 2 - 2k^{-1}, \qquad A_2 = 1,\ A_3 = 4/3, \quad \dots \quad A_\infty = 2$$

$$n_k = \#\{a_j : A_k \le \alpha_j < A_{k+1}\}, \qquad n_\infty = \#\{a_j : \alpha_j \ge 2\}$$

$$A(\Delta) = \sum_{2 \le k \le \infty} n_k A_k = n_2 + \frac{4}{3} n_3 + \frac{3}{2} n_4 + \dots + 2n_\infty$$

Singularities $\Delta_{reg} = \{a_j : \alpha_j = A_k\}$ are regularizable (Knauf 1987). $A(\Delta) \le S(\Delta)$ and $S(\Delta) = A(\Delta)$ iff $\Delta = \Delta_{reg}$.

- If all singularities are weak, $A(\Delta) = 0$.
- If all singularities are Newtonian, $A(\Delta) = n$.
- If all singularities are strong, $A(\Delta) = 2n$.

Theorem (B-Kozlov 2017)

Let M be a closed manifold, there are no gyroscopic forces, and $A(\Delta) > 2\chi(M)$. Then for $h > \max V$ there is a compact invariant set with positive topological entropy on $\{H = h\}$.

Examples

- $M = \mathbb{T}^2$, $\chi(\mathbb{T}^2) = 0$. The theorem works if there is a nonweak singularity. We do not know if the existence of a weak singularity on $\mathbb{T}^2$ always implies chaotic behavior.
- $M = S^2$, $\chi(S^2) = 2$. The theorem works for:
  - $n \geq 5$ singularities with $\alpha_j \geq 1$,
  - $n \geq 4$ singularities with $\alpha_j \geq 4/3$,
  - $n \geq 3$ singularities with $\alpha_j \geq 3/2$.
  - 3 singularities with $\alpha_j \geq 1$ and the 4th with $\alpha_4 \geq 4/3$.
- For $n = 4$ Newtonian singularities on S^2 the system may be integrable on an energy level $H = h > \max V$.

Noncompact M

Take a compact domain $D \subset M$ containing Δ. The gyroscopic form is exact on D:

$$\Gamma = d\omega, \quad \omega = \langle w(q), dq \rangle.$$

Trajectories on $\{H = h\}$ are geodesics of the Jacobi metric

$$g_h(q, \dot q) = \sqrt{2(h - V(q))}\|\dot q\| + \langle w(q), dq \rangle.$$

g_h is positive definite in D if

$$h > \max_{q \in D}(V(q) + \|w(q)\|^2/2).$$

The domain D is geodesically convex with respect to g_h if

$$\langle \nabla V, \nu \rangle + 2\kappa(h - V) - \|\Gamma\|\sqrt{2(h - V)} \geq 0 \quad \text{on } \partial D.$$

ν – the inner unit normal to ∂D; κ – the geodesic curvature.

Main theorem

Theorem

Let $D \subset M$ be a compact domain containing Δ. Suppose D is geodesically convex for energy h. If $A(\Delta) > 2\chi(D)$, there is a compact invariant set with positive topological entropy on $\{H = h\} \cap T^*(D \setminus \Delta)$.

Theorem

If $A(\Delta) > 2$, the generalized n-center problem in $\mathbb{R}^2$ has a compact chaotic invariant set on any energy level $H = h > \sup V$.

A weaker result was proved in (Castelli 2017).

Proof: there exists a convex for energy h compact set $D \subset \mathbb{R}^2$ containing all nonweak singularities. □

- If $\alpha_j \geq 1$, the theorem works for $n \geq 3$.
- Works for $n = 2$ if $\alpha_1 \geq 1$ and $\alpha_2 \geq 4/3$.
- Works for the plane restricted circular $n + 1$ body problem with $n \geq 3$.

Convexity properties

Lemma

For $0 < \alpha_j < 2$ *and small* $\varepsilon > 0$ *the ball*

$$B_j = B(a_j, \varepsilon) = \{q : d(x, a_j) \le \varepsilon\}, \quad a_j \in \Delta.$$

is geodesically convex in the Jacobi metric. For $\alpha_j > 2$, *the complement of* B_j *is geodesically convex.*

Known to the founders of celestial mechanics. We can remove strong singularities replacing D by a geodesically convex $D' = D \setminus \cup_{\alpha_j > 2} B_j$.

Trajectories with energy h are extremals of the Jacobi length

$$J(\gamma) = \int g_h(\gamma(t), \dot{\gamma}(t))\, dt.$$

Distance function

$$\rho(x, y) = \inf\{J(\gamma) : \gamma \in C^1([0, 1], D \setminus \Delta),\ \gamma(0) = x,\ \gamma(1) = y\}.$$

Lemma

Any points $x, y \in D$ can be joined by a minimizer such that $J(\gamma) = \rho(x, y)$. The minimizer is a chain $\gamma = \gamma_1 \cdots \gamma_k$ of trajectories joining pairs of points in Δ.

In order to get noncollision trajectories we need to show that the minimizers do not pass through Δ.

Cone property

The next result is the crucial step in the proof. Suppose all singularities are weak or moderate: $0 < \alpha_j < 2$.

Lemma

For small $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for any $x, y \in \partial B(a_j, \varepsilon)$, the minimizer joining x, y does not enter $B(a_j, \delta)$.

Corollary

Let $\pi : \tilde{D} \to D$ be the universal covering and $\tilde{\Delta} = \pi^{-1}(\Delta)$. A minimizer joining any points $x, y \in \tilde{D} \setminus \tilde{\Delta}$ does not pass though $\tilde{\Delta}$.

Weak main theorem

Proposition

If all singularities are weak and $\chi(D) < 0$, the geodesic flow of the Jacobi metric on $D \setminus \Delta$ has a compact chaotic invariant set.

Proof. Modify the Jacobi metric g_h in every ball $B_j = B(a_j, \delta)$ replacing it by a smooth Riemannian metric $\tilde{g}$ on D such that $\tilde{g} = g_h$ on $D \setminus \cup B_j$ and $\tilde{g} \geq g_h$ in $\cup B_j$. Since $\chi(D) < 0$, there is a compact chaotic invariant set of minimizing geodesics of the metric $\tilde{g}$ (Morse, Dinaburg, Kozlov). By the cone property, such geodesics can not pass through B_j, so they are geodesics of the Jacobi metric. □

The main theorem follows from the proposition via regularization: we make singularities weak while complicating topology.

Generalized Levi-Civita regularization

Choose local conformal coordinate $z \in \mathbb{C}$ so that $a_j = \{z = 0\}$.

$$V(z) = -\frac{f(z)}{|z|^\alpha} + U(z), \qquad \alpha = \alpha_j, \quad f(0) > 0.$$

The transformation $z = w^\beta$ replaces the Jacobi metric g_h by $\tilde{g}$:

$$\begin{aligned} g_h &= 2\sqrt{g(z)(f(z)|z|^{-\alpha} + h - U(z))}|\dot{z}| \\ \tilde{g} &= 2\beta\sqrt{g(w^\beta)(f(w^\beta)|w|^{-\tilde{\alpha}} + (h - U(w^\beta))|w|^{2(\beta-1)})}|\dot{w}|. \end{aligned}$$

Singularity of order α is replaced by a singularity of order

$$\tilde{\alpha} = \alpha\beta - 2(\beta - 1).$$

If $\alpha = A_k = 2 - 2/k$, and $\beta = k \in \mathbb{N}$, the transformation $\phi_k(w) = w^k$ regularizes the singularity (Knauf 1987):

$$\tilde{\alpha} = 2 - k(2 - \alpha) = 0.$$

The classical Levi-Civita transformation for $k = 2$.

Lemma

If $A_k < \alpha_j < A_{k+1}$, the transformation ϕ_k replaces the singularity a_j by a weak singularity of order $\tilde{\alpha}_j = 2 - k(2 - \alpha_j)$.

For Jacobi singularities $\tilde{\alpha}_j = \alpha_j = 2$: regularization fails.

Global regularization

Theorem

There exists a surface X with boundary, a K-sheet smooth covering $\phi : X \to D \setminus (\Delta_{jac} \cup \Delta_{strong})$ branched over the set $\Delta_{newt} \cup \Delta_{mod}$, and a smooth Riemannian metric on X such that:

- *Projections to D of minimal geodesics on the universal covering of X are trajectories with energy $H = h$ having no collisions with Δ, except maybe with regularizable singularities Δ_{reg}.*
- *The Euler characteristics*

$$\chi(X) = K\big(\chi(D) - \frac{1}{2}A(\Delta)\big) < 0.$$

Proof

For simplicity let $D \subset \mathbb{R}^2 \cong \mathbb{C}$ and $\Delta_{strong} = \Delta_{jac} = \emptyset$.

$\Delta = \Delta_{mod} \cup \Delta_{newt} = \cup_{i=1}^m \Delta_{k_i}, \qquad \Delta_{k_i} = \{a_j : A_k \le \alpha_j < A_{k+1}\}.$

$$X = \{(z, w_1, \dots, w_m) \in D \times \mathbb{C}^m : w_i^{k_i} = \prod_{a_j \in \Delta_{k_i}} (z - a_j)\}$$

is a smooth complex curve. The projection

$$\pi : X \to D, \qquad \pi(z, w_1, \dots, w_m) = z,$$

is a covering with the number of sheets

$$\#\pi^{-1}(z) = \prod_{i=1}^m k_i = K, \qquad z \in D \setminus \Delta.$$

The covering $\pi : X \to D$ is branched over Δ:

$$\#\pi^{-1}(a_j) = \frac{K}{k_i}, \qquad a_j \in \Delta_{k_i}.$$

Near $q \in \pi^{-1}(\Delta_{k_i})$, π is the generalized Levi-Civita transformation of order $\nu(q) = k_i$.
By the Riemann–Hurwitz formula,

$$\begin{aligned}
\chi(X) &= K\chi(D) - \sum_{i=1}^{m} \sum_{q\in\pi^{-1}(\Delta_{k_i})} (\nu(q) - 1) \\
&= K\chi(D) - \sum_{i=1}^{m} n_{k_i} \frac{K}{k_i}(k_i - 1), \qquad n_{k_i} = \#\Delta_{k_i}, \\
&= K\chi(D) - \frac{K}{2}\sum_{i=1}^{m} n_{k_i} A_{k_i} = K(\chi(D) - \frac{1}{2}A(\Delta)) < 0.
\end{aligned}$$

End of the proof

$\pi : X \setminus \pi^{-1}(\Delta) \to D \setminus \Delta$ is a nonbranched covering. Lift the Jacobi metric g_h on $D \setminus \Delta$ to a metric on $X \setminus \pi^{-1}(\Delta)$.
The singularity a_j disappears if $\alpha_j = A_{k_i}$ and becomes weak if $A_{k_i} < \alpha_j < A_{k_{i+1}}$.
Since $\chi(X) < 0$, by the weak main theorem, the geodesic flow on X has a compact chaotic invariant set.
When D is a plane domain and there are no Jacobi singularities, the proof is finished.
For Jacobi singularities a different argument is needed.

Spatial n-center problem

Our results can be partly extended to the spatial n-center problem. For $n \geq 3$ Newtonian singularities in $\mathbb{R}^3$ the existence of a chaotic invariant set was proved using global KS regularization (B-Negrini 2000).

The regularization replaces the configuration space $M = \mathbb{R}^3$ by the 4-dimensional manifold $\hat{M}_n$:

$$\hat{M}_n = (S^2 \times \mathbb{R}^2)\#\underbrace{(S^2 \times S^2)\#\ldots\#(S^2 \times S^2)}_{k-2}, \quad n = 2k$$

$$\hat{M}_n = \mathbb{R}^4\#\underbrace{(S^2 \times S^2)\#\ldots\#(S^2 \times S^2)}_{k-1}, \quad n = 2k+1.$$

For $n \geq 3$ this is a rationally hyperbolic 4-manifold. Then a modification of Gromov's theorem can be used to prove positive topological entropy.

Generalized spatial n-center problem

Proposition

If there are $n \geq 3$ singularities with $\alpha_j > 1$, global KS regularization gives a system on $\hat{M}_n$ with weaker singularities $0 < \tilde{\alpha}_j < \alpha_j$.

Then we hope that a modification of Gromov's theorem can be applied to obtain a chaotic invariant set. The problem is that, contrary to the 2-dimensional case, we can't exclude that non-minimizing chaotic trajectories enter weak singularities.

Conjecture

Let $B_k = 2 - 2^{k-1}, \quad m_k = \#\{a_k : B_k \leq \alpha_j < B_{k+1}\}$. *If*

$$B(\Delta) = \sum_{1 \leq k \leq \infty} m_k B_k > 2,$$

then the generalized n center problem in $\mathbb{R}^3$ *has positive topological entropy on energy levels* $H = h > \sup V$.

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Thank you for your attention!