THE THREE-BODY PROBLEM IN SHAPE SPACE

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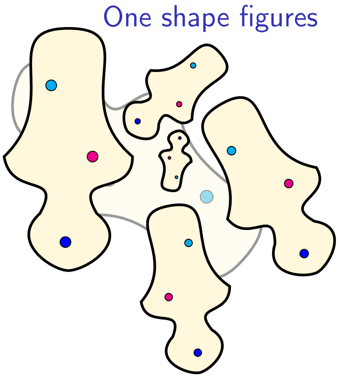
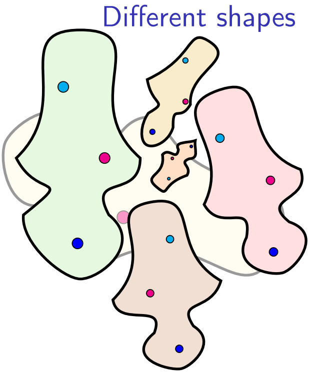
What is it about?

- What is SHAPE (space, sphere etc)?
- Formulas and others
- Hamiltonian, energy integral and region of possible motion
- Topology of possible motion space
- Invariant configurations
- Periodic orbits
- Periodic orbits in shape space
- Regularization
- Degenerate orbits and chaos

What is SHAPE

Two planar figures have the same shape if one figure is exactly superposed on the other by some translation, rotation and scaling.



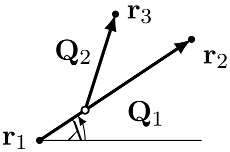


Note 1. For TBP the configuration of bodies is always triangle.

Note 2. The size is important for TBP as well.

Shape space, I. Reduction by translation ($\mathbb{R}^9 \to \mathbb{R}^6$)

$\mathbb{R}^6 \to \mathbb{R}^4$



$$\begin{aligned} \mathbf{Q}_1 &= \mathbf{r}_2 - \mathbf{r}_1, \\ \mathbf{Q}_2 &= \mathbf{r}_3 - \tfrac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}, \end{aligned}$$

$$m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + m_3\mathbf{x}_3 = 0.$$

In Jacobie coordinates:

$$\begin{aligned} r_{12} &= \left|\mathbf{Q}_1\right|, \\ r_{13} &= \left|\mathbf{Q}_2 + \frac{m_2}{m_1+m_2}\mathbf{Q}_1\right|, \\ r_{23} &= \left|\mathbf{Q}_2 - \frac{m_1}{m_1+m_2}\mathbf{Q}_1\right|. \end{aligned}$$

$$T = \tfrac{1}{2}(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2 + m_3\dot{\mathbf{r}}_3^2) = \tfrac{1}{2}\left(\mu_1\dot{\mathbf{Q}}_1^2 + \mu_2\dot{\mathbf{Q}}_2^2\right),$$

$$L = T(\dot{\mathbf{Q}}_1, \dot{\mathbf{Q}}_2) - V(\mathbf{Q}_1, \mathbf{Q}_2),$$

$$J = \mu_1\mathbf{Q}_1 \times \dot{\mathbf{Q}}_1 + \mu_2\mathbf{Q}_2 \times \dot{\mathbf{Q}}_2.$$

Here $\mu_1 = m_1m_2/(m_1+m_2)$, $\mu_2 = m_3(m_1+m_2)/(m_1+m_2+m_3)$.

Shape space, II. Reduction by rotation

Reducton by rotation (Hopf mapping):

$$\mathbb{R}^4 \to \mathbb{R}^3 : \mathcal{S}^1 \hookrightarrow \mathcal{S}^3 \to \mathcal{S}^2$$
$$(\mathcal{S}^3 \hookrightarrow \mathcal{S}^7 \to \mathcal{S}^4).$$

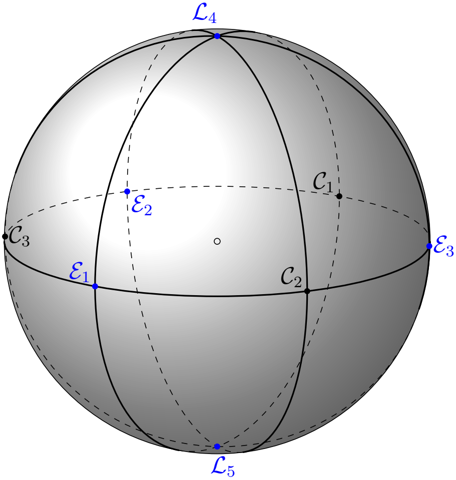
$$\begin{aligned}
\xi_1 &= \mu_1|\mathbf{Q}_1|^2 - \mu_2|\mathbf{Q}_2|^2, \\
\xi_2 &= 2\sqrt{\mu_1\mu_2}\ \mathbf{Q}_1 \cdot \mathbf{Q}_2, \\
\xi_3 &= 2\sqrt{\mu_1\mu_2}\ \mathbf{Q}_1 \times \mathbf{Q}_2,
\end{aligned}$$

Moment of inertia:

$$\begin{aligned}
I &= m_1|\mathbf{r}_1|^2 + m_2|\mathbf{r}_2|^2 + m_3|\mathbf{r}_3|^2 = m_1m_2r_{12}^2 + m_1m_3r_{13}^2 + m_2m_3r_{23}^2 = \\
&= \mu_1|\mathbf{Q}_1|^2 + \mu_2|\mathbf{Q}_2|^2 = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.
\end{aligned}$$

Shape space, III

Shape space, $\Xi = (\xi_1, \xi_2, \xi_3)$, is the space of congruent triangles.
Shape sphere, $S^2 \subset \Xi$, is the space of similar triangles.



If $\mathbf{r}_i(t)$, $i = 1, \ldots, N$, is solution of N-body problem, the followimg expression gives the solution as well

$$\boldsymbol{\rho}_i(t) = \lambda \mathbf{r}_i(\lambda^{-3/2} t)$$

In addition

$$\begin{array}{rcl} \dot{\boldsymbol{\rho}}_i & = & \lambda^{-1/2} \mathbf{v}_i(\lambda^{-3/2} t) \\ h' & = & h/\lambda \end{array}$$

Expressions for mutual distances

$$
\begin{aligned}
r_{12}^2 &= \frac{m_1 + m_2}{2m_1m_2}(\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} + \xi_1) \\
r_{13}^2 &= \frac{m_1 + m_3}{2m_1m_3}\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \\
&\quad + \frac{m_2m_3 - m_1(m_1 + m_2 + m_3)}{2m_1m_3(m_1 + m_2)}\xi_1 + \frac{\sqrt{m_1m_2m_3(m_1 + m_2 + m_3)}}{m_1m_3(m_1 + m_2)}\xi_2 \\
r_{23}^2 &= \frac{m_2 + m_3}{2m_2m_3}\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \\
&\quad + \frac{m_1m_3 - m_2(m_1 + m_2 + m_3)}{2m_2m_3(m_1 + m_2)}\xi_1 - \frac{\sqrt{m_1m_2m_3(m_1 + m_2 + m_3)}}{m_2m_3(m_1 + m_2)}\xi_2
\end{aligned}
$$

$$
\begin{aligned}
T &= \tfrac{4J^2+\dot{\xi}_1^2+\dot{\xi}_2^2+\dot{\xi}_3^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}, \\
J &= \mu_1 \mathbf{Q}_1 \times \dot{\mathbf{Q}}_1 + \mu_2 \mathbf{Q}_2 \times \dot{\mathbf{Q}}_2, \\
&= \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\tfrac{d\lambda}{dt} + \tfrac{\xi_2 \frac{d\xi_3}{dt} - \xi_3 \frac{d\xi_2}{dt}}{2(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)}
\end{aligned}
$$

Clearly, the angular momentum J is conjugated to the angle λ. If we know ξ_1, ξ_2, ξ_3, we can obtain this angle λ from quadrature:

$$
\lambda(t) = \int_0^t \frac{\partial R}{\partial J}\, d\tau = \int_0^t \frac{J - \frac{\xi_2\dot{\xi}_3 - \xi_3\dot{\xi}_2}{2\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1}}{\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}} d\tau.
$$

The system is conservative, so we have energy integral:

$$
T - U = \frac{4J^2+\dot{\xi}_1^2+\dot{\xi}_2^2+\dot{\xi}_3^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}} - U(\xi_1,\xi_2,\xi_3) = h
$$

The region of possible region

Proposition (Arnold et all, 1.7)

We have the inequality: $J^2 \leq 2IT$

$$J^2 = \left|\sum m_i(\mathbf{r}_i \times \mathbf{v}_i)\right|^2 \leq \left(\sum m_i |\mathbf{r}_i|\,|\mathbf{v}_i|\right)^2 \leq \left(\sum m_i \mathbf{r}_i^2\right)\left(\sum m_i \mathbf{v}_i^2\right) = I \cdot 2T.$$

Constraints (the kinetic integrals)

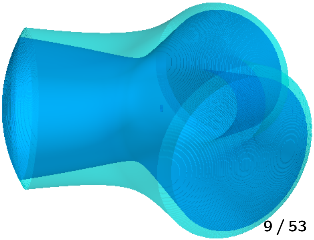
$$\Gamma = \left\{\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n) \in \mathbb{R}^{3n} : \sum m_i \mathbf{r}_i = 0\right\}.$$

In planar three body problem (Proposition 1.8)

$$B_{J,h} = \left\{\mathbf{r} \in \Gamma : U + \frac{J^2}{2I} \leq h\right\},$$

in spacial one

$$B_{J,h} \subset \left\{\mathbf{r} \in \Gamma : U + \frac{J^2}{2I} \leq h\right\} \subset \Gamma.$$



Energy integral, Sundman inequality and zero velocity surface

$$\frac{J^2}{2I} - U(\xi_1, \xi_2, \xi_3) - h \leq$$

$$\frac{J^2}{2I} + \dot{I}^2/(8I) - U(\xi_1, \xi_2, \xi_3) - h \quad \leq$$

$$\frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} - U(\xi_1, \xi_2, \xi_3) - h = 0,$$

$$\frac{J^2}{2\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} - U(\xi_1, \xi_2, \xi_3) = h,$$

$$U(\xi_1, \xi_2, \xi_3) = \frac{1}{\sqrt[4]{\xi_1^2 + \xi_2^2 + \xi_3^2}}\, D(\varphi, \theta).$$

$m_1 = m_2 = m_3 = 1$: $O\xi_3$ и $O\xi_1$

$$
\begin{aligned}
J = 0: \quad U(0,0,\xi_3) = &\quad 3/\sqrt{\xi_3} = -h = \frac{1}{2}, \quad \to \quad \xi_3 = 36 \\
U(\xi_1,0,0) = &\quad 1/\sqrt{2\xi_1} + 2/\sqrt{\xi_1/2}, \quad \to \quad \xi_1 = 50.
\end{aligned}
$$

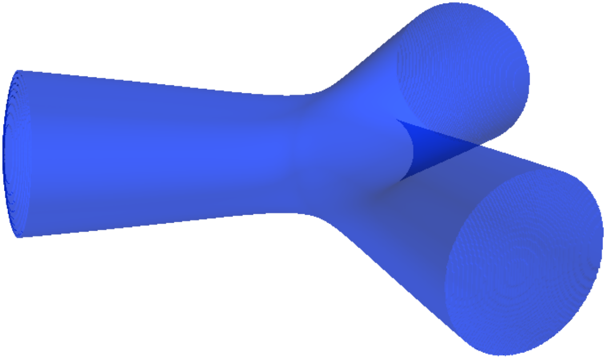
$$
J \neq 0: \qquad U(0,0,\xi_3) - \frac{J^2}{2\xi_3} = 3/\sqrt{\xi_3} - \frac{J^2}{2\xi_3} = \frac{1}{2} \quad \to
$$

$$
\xi_3 \in \left[\left(3 - \sqrt{9 - J^2}\right)^2, \left(3 + \sqrt{9 - J^2}\right)^2\right].
$$

$$
U(\xi_1,0,0) - \frac{J^2}{2\xi_1} = 1/\sqrt{2\xi_1} + 2/\sqrt{\xi_1/2} - \frac{J^2}{2\xi_1} = \frac{1}{2} \quad \to
$$

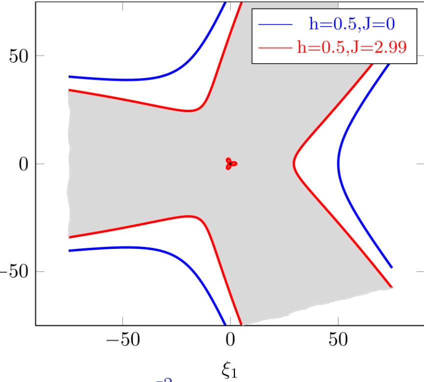
$$
\xi_1 \in \left[\left(5 - \sqrt{25 - 2\,J^2}\right)^2/2, \left(5 + \sqrt{25 - 2\,J^2}\right)^2/2\right].
$$

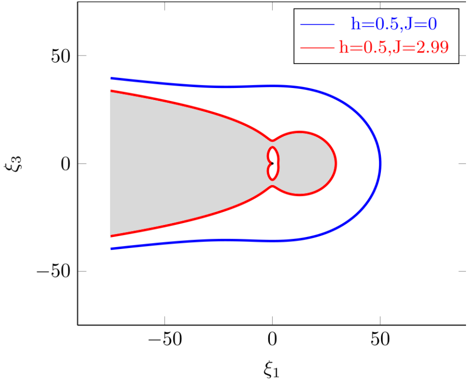
Zero velocity surface at $J = 0$ ($h = -\frac{1}{2}$)



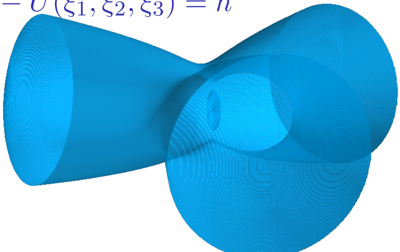
$$U(\xi_1, \xi_2, \xi_3) = -h$$

Zero velocity surface at $J = 2.99$)

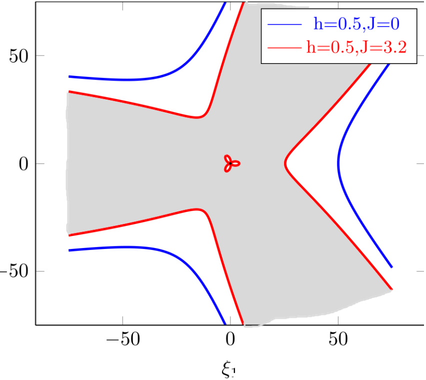


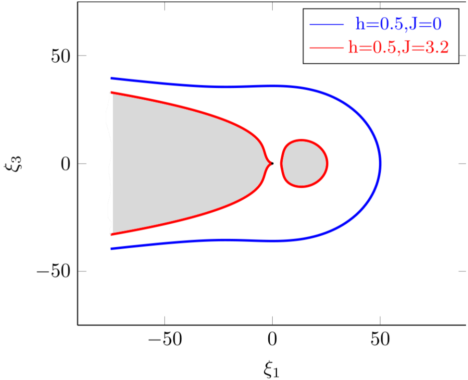


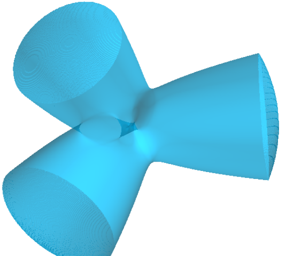
$$\frac{J^2}{2\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}} - U(\xi_1, \xi_2, \xi_3) = h$$



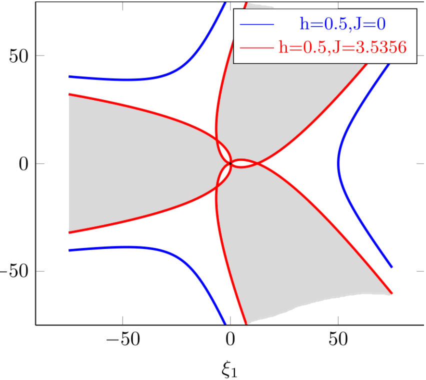
Zero velocity surface at $J = 3.2$)

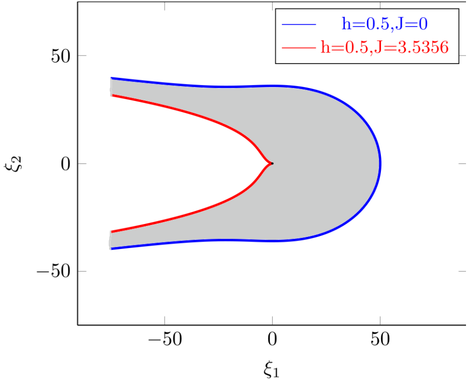


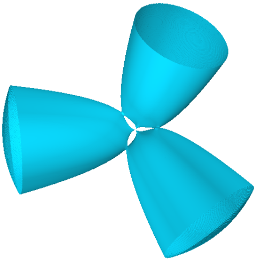




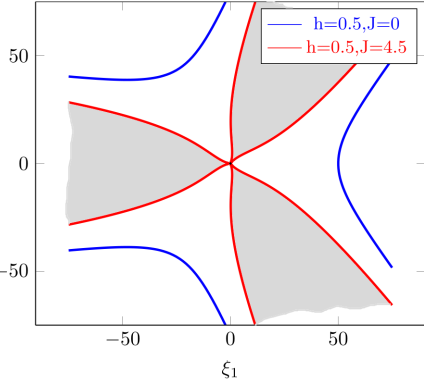
Zero velocity surface at $J = 3.53$)

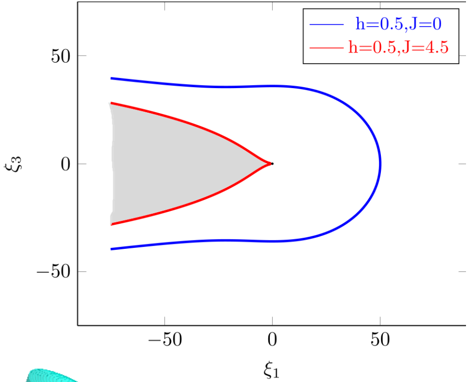


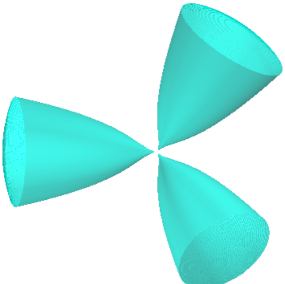




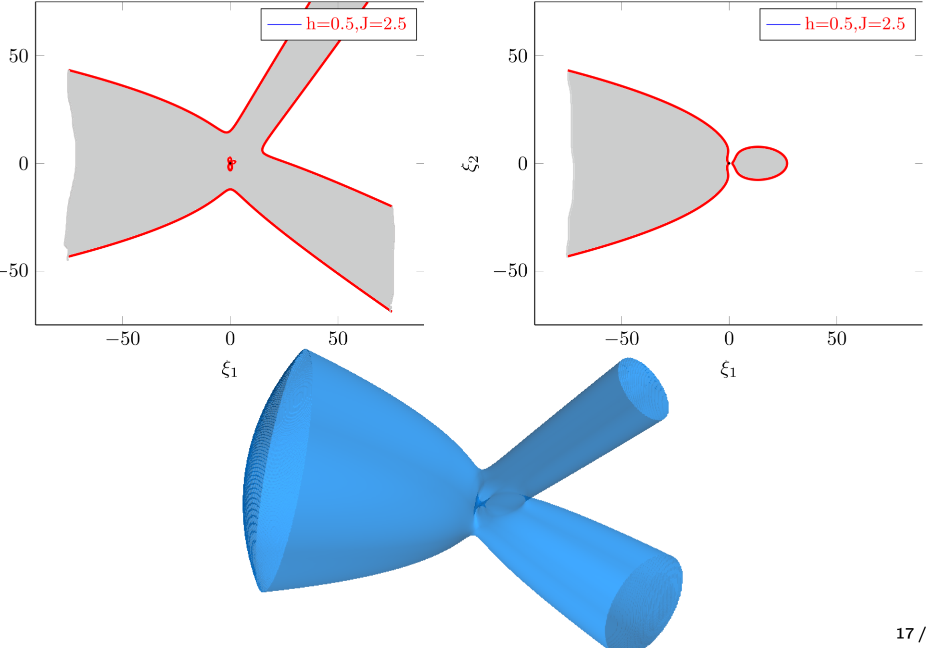
Zero velocity surface at $J = 4.5$ $(J > 5/\sqrt{2})$



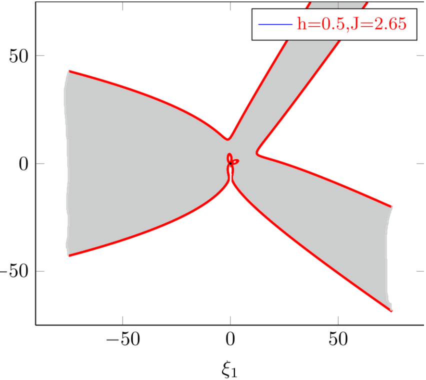


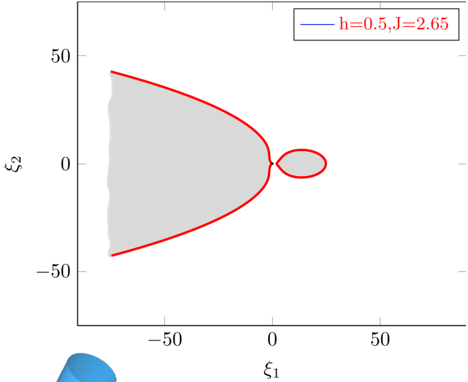


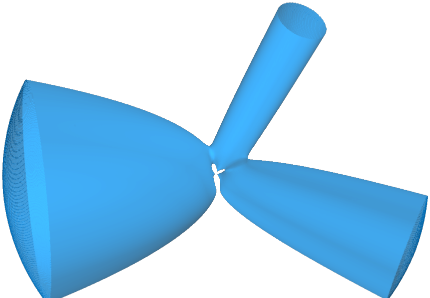
$m_1 = 2m_2 = 4m_3 = 12/7,\ h = -1/2,\ J = 2.5$



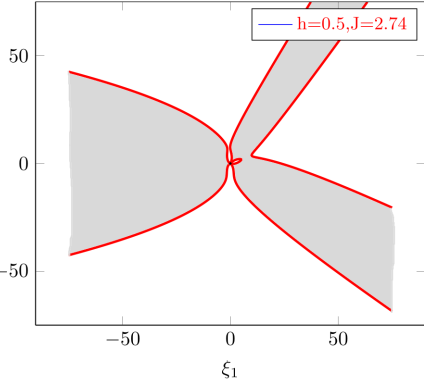
$m_1 = 2m_2 = 4m_3 = 12/7,\ J = 2.65$

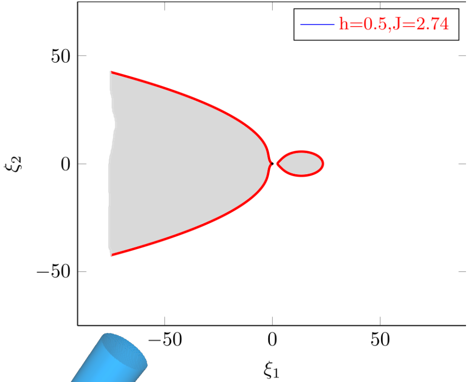


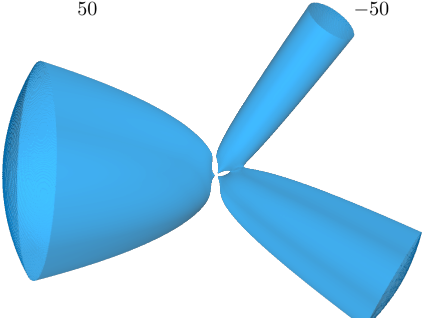




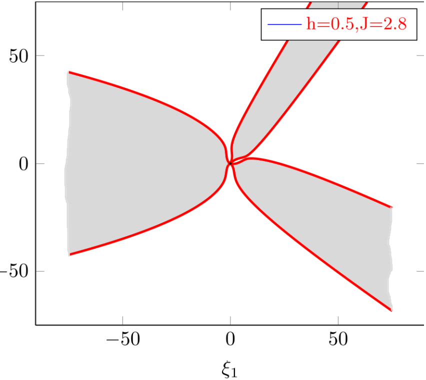
$m_1 = 2m_2 = 4m_3 = 12/7,\ J = 2.74$

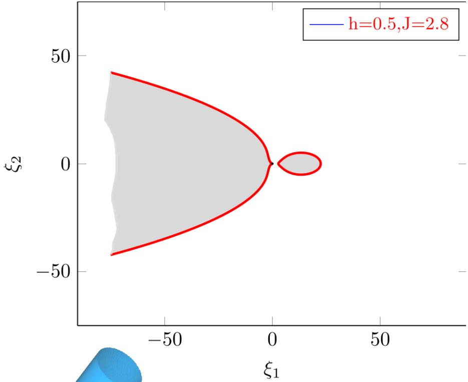


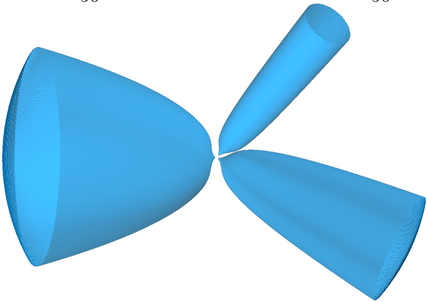




$m_1 = 2m_2 = 4m_3 = 12/7,\ J = 2.8$







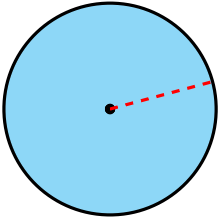
Topology of possible motion space

1. The available is bounded by a surface with three branches. Motion is possible inside this space with the exception of the punctured point, the origin of coordinates. As J grows the outer surface with three branches decreases, and the punctured point becomes a surface whose cross-section with the equator plane resembles a trefoil, and the cross-section with the meridian plane has the shape «frigole»,
2. with growth of J, the outer and inner surfaces join together and a hole forms in the outer surface,
3. with further growth of J, two branches are first separated, the space of possible motion is still connected,
4. but as J increases, one branch separates from the other two, and finally,
5. beginning from a certain value of J, we have three separate areas of possible movement.

Zero velocity surface (2BP)

The energy integral

$$\begin{aligned} T - V &= \frac{\dot{\mathbf{r}}^2}{2} - \frac{1}{r} = h \\ r &\leq -\frac{1}{h}, \quad h < 0. \end{aligned}$$



Adding the angular momentum integral

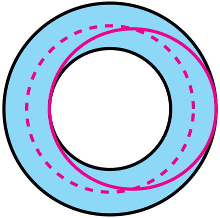
$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{J} = r^2 \dot{\theta}$$

yields

$$\frac{2}{r} + 2h - \frac{J^2}{r^2} \geq 0$$

or

$$\begin{cases} r_{\text{min}} \leq r \leq r_{\text{max}} & \text{if } h < 0, \\ r > r_{\text{min}} & \text{if } h \geq 0 \end{cases}$$



Cartesian relative or baricentric coordinate system

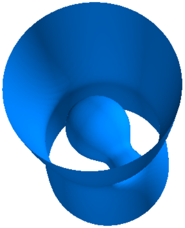
Zero velocity surface (Hill, CR3BP)

The Hill surface, depending on the Jacobi constant, consists of 1, 2, or 3 pieces. One or two pieces are non-compact.

1

2

3



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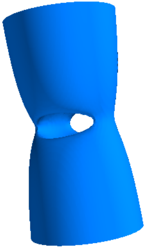

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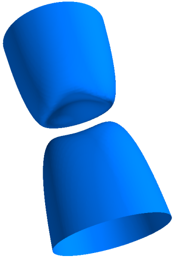
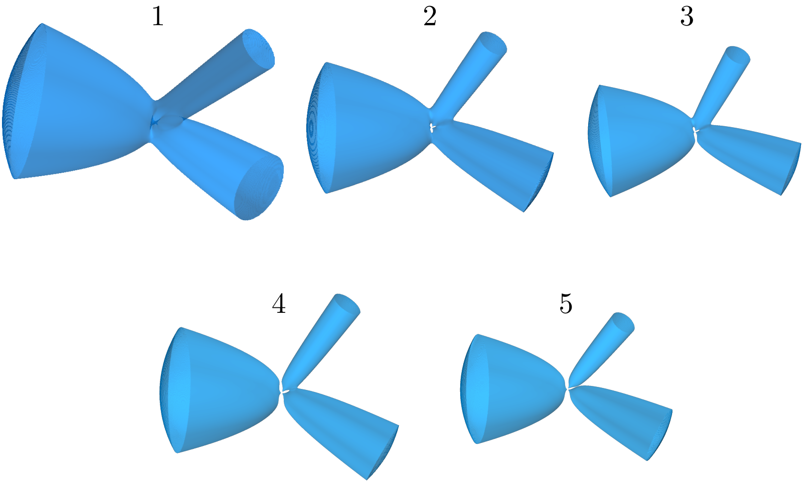
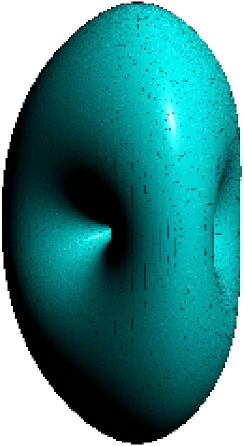


Zero velocity surface (general three-body problem)



Final motions, hierarchical systems

Inner surface



Shape space, spherical coordinates

$$\begin{aligned}\xi_1 &= \rho\cos\varphi\cos\theta,\\ \xi_2 &= \rho\sin\varphi\cos\theta,\\ \xi_3 &= \rho\sin\theta,\end{aligned}$$

$$\begin{aligned}r_{12}^2 &= \frac{m_1+m_2}{2m_1m_2}\,\rho(1+\cos\varphi\cos\theta),\\ r_{13}^2 &= \frac{m_1+m_3}{2m_1m_3}\,\rho(1-\cos(\varphi-\varphi_{13})\cos\theta),\\ r_{23}^2 &= \frac{m_2+m_3}{2m_2m_3}\,\rho(1-\cos(\varphi-\varphi_{23})\cos\theta).\end{aligned}$$

$$V(\rho,\theta,\varphi) = \frac{1}{\sqrt{\rho}}\left(\frac{\nu_{12}}{\sqrt{1+\cos\theta\cos\varphi}} + \frac{\nu_{13}}{\sqrt{1-\cos\theta\cos(\varphi-\varphi_{13})}} + \frac{\nu_{23}}{\sqrt{1-\cos\theta\cos(\varphi-\varphi_{23})}}\right) = \frac{1}{\sqrt{\rho}}\,D(\theta,\varphi),$$

Invariant configurations ($\theta = \mathrm{const}, \varphi = \mathrm{const}$):

$$\begin{aligned}\frac{\partial D(\varphi,\theta)}{\partial\theta} &= 0,\\ \frac{\partial D(\varphi,\theta)}{\partial\varphi} &= 0\end{aligned}$$

Motion with invariant configuration

$$V = \frac{C_1}{\sqrt{\rho}} \,.$$

Lagrange–Jacobie identity

$$\ddot{I} = 2\left(\frac{C_1}{\sqrt{I}} + 2h\right),$$

reduces to ($r^2 = I, \frac{dt}{d\tau} = r = \sqrt{I}$) к

$$r'' = 2hr + C_1.$$

The solution is ($h < 0$)

$$r = \sqrt{I} = A\cos(n\tau - \vartheta) - \frac{C_1}{2h}, \quad n = \sqrt{-2h}.$$

Periodic orbits

Finite symmetries. For planar 3BP there are 10 finite symmetry groups only. Three of them are considered here for illustration the trajectories in shape space: the dihedral group (D_{12}, simple choreography), $2-1$ choreographies and Line symmetry.

Periodic orbits are searched as minimizator of action functional

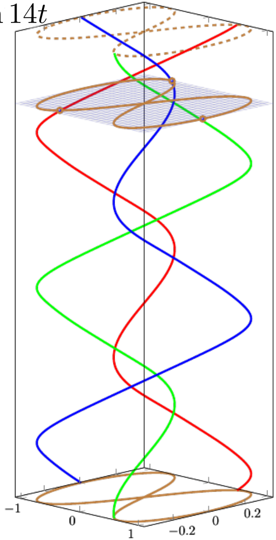
$$\mathcal{A} = \int_{t_1}^{t_2} L(\mathbf{q}_i, \dot{\mathbf{q}}_i, t).$$

in the form

$$\begin{aligned} x_j(t) &= C_x^0 + \sum_{i=1} C_{xi}^j \cos\, it + S_{xi}^j \sin\, it \\ y_j(t) &= C_y^0 + \sum_{i=1} C_{y_i}^j \cos\, it + S_{y_i}^j \sin\, it, \end{aligned}$$

Figure-Eight

$$
\begin{aligned}
x(t) &= 1.0959\sin t - 0.0253\sin 5t - 0.0058\sin 7t \\
&\quad +0.0004\sin 11t + 0.0001\sin 13t \\
y(t) &= 0.3373\sin 2t + 0.0557\sin 4t - 0.0030\sin 8t \\
&\quad -0.0008\sin 10t + 0.0001\sin 14t
\end{aligned}
$$



Numerically found by Cr.Moore in “Braids in classical dynamics”, Phys. Rev. Lett. 1993, **70**.

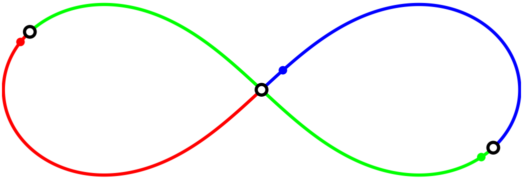
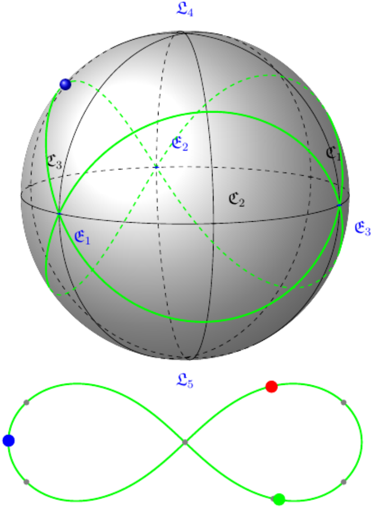
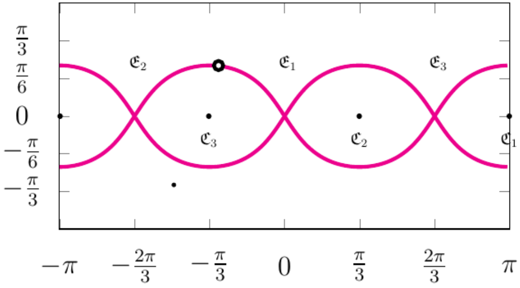


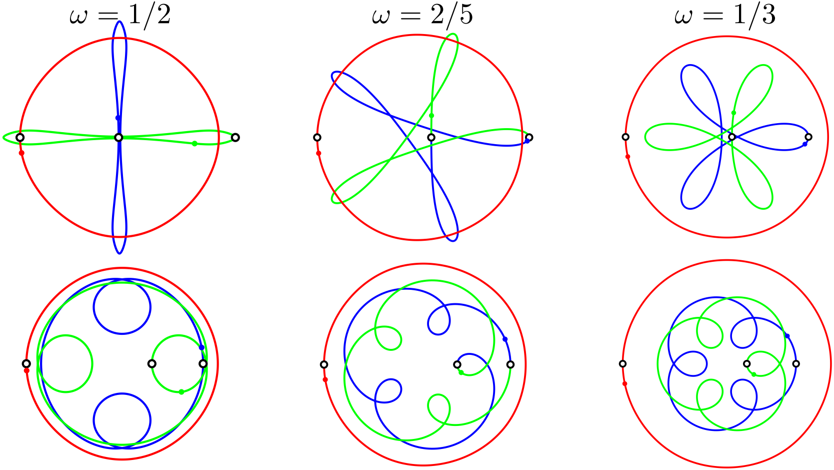
Figure-Eight on the shape sphere





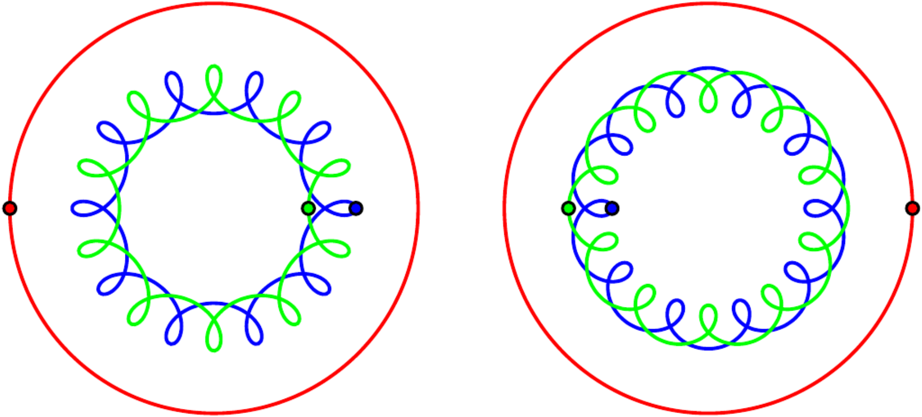
$2 - 1$ choreography

Two bodies of equal mass move along the same trajectory with a phase lag π. A group have a Rotating Circle Property. An orbit in an inertial system is a minimizer found in a system rotating with angular velocity ω.



Tight binaries

$$\omega = 1/3,\ k = 5$$

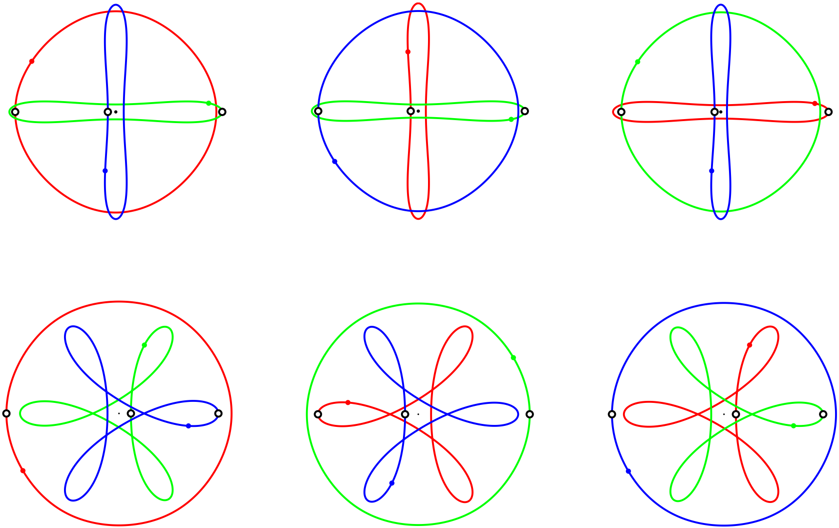


$2-1$ symmetry choreography

$m_1 = m_2 = 0.95,\ m_3 = 1.1$					
A	E	$\lvert C\rvert$	ω	$[I_{\mathrm{min}}, I_{\mathrm{max}}]$	St
10.61083	-0.562922	1.73204	1/5	[13.520,18.706]	+
11.87886	-0.630193	1.34061	1/3	[7.646,7.695]	+
12.41405	-0.658586	1.22094	2/5	[6.446,6.518]	+
12.43822	-0.850687	3.17929	1/5	[13.037,13.062]	+
13.13826	-0.697007	1.09433	1/2	[5.463,5.580]	+
14.90941	-0.790968	2.76171	1/3	[6.779,6.847]	+
16.03507	-0.850687	2.61695	2/5	[5.352,5.457]	+
16.57031	-0.879082	2.44831	1/3	[3.869,3.957]	-
17.61955	-0.934746	2.43060	1/2	[3.967,4.154]	-
19.78460	-1.049610	1.57727	1/3	[6.501,6.503]	+
21.89957	-1.161810	2.58582	1/3	[6.441,6.443]	+
25.74992	-1.366082	1.65989	1/3	[6.380,6.381]	+
27.53447	-1.460752	2.51159	1/3	[6.362,6.363]	+
$m_1 = m_2 = 1.05,\ m_3 = 0.9$					
12.20094	-0.647280	0.98928	1/3	[7.276,7.323]	+
15.79177	-0.837779	2.68412	1/3	[6.275,6.341]	+
16.61662	-0.881539	2.33447	1/3	[3.779,3.943]	-
Figure-eight: $m_1 = m_2 = m_3 = 1.0$					
24.37193	-1.29297	0	-	[1.973,1.982]	+

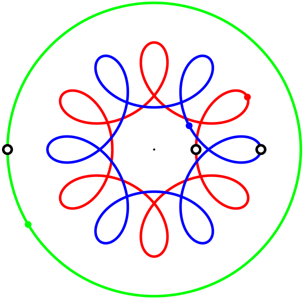
Line symmetry orbits

Three orbits with linear symmetry corresponding to cyclic permutation of masses m_1, m_2, m_3 ($\omega = 1/2$ и $\omega = 1/3$).

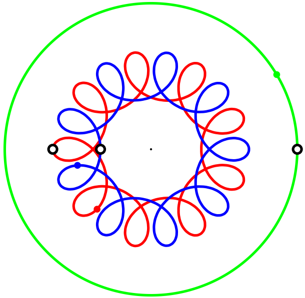


Orbits with Tight binaries

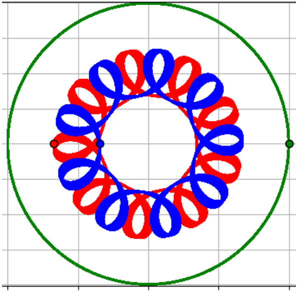
$\omega = 1/3$



$k = 2$



$k = 3$

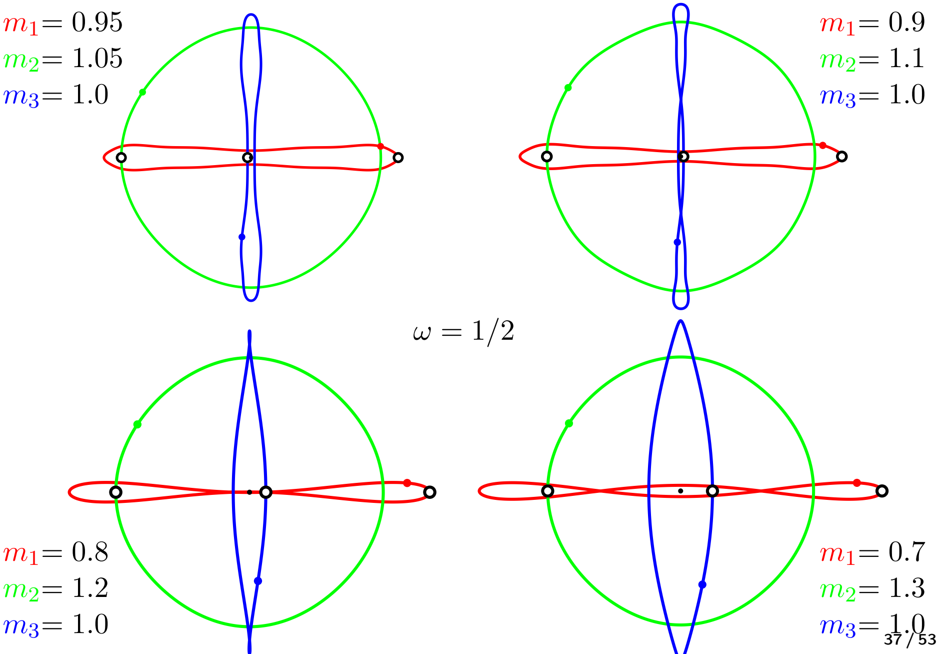


Numerical integration
500 periods

Line symmetry orbits

$m_1 = 0.99,\ m_2 = 1.01,\ m_3 = 1.0$					
A	E	$\|J\|$	ω	$[I_{\text{min}}, I_{\text{max}}]$	St
11.42286	-0.606002	1.36301	1/4	[10.095,10.108]	+
12.04740	-0.639135	1.19429	1/3	[7.508,7.557]	+
12.06332	-0.639979	1.17690	1/3	[7.489,7.538]	+
12.07962	-0.640844	1.15915	1/3	[7.471,7.520]	+
13.15385	-0.697833	0.92132	1/2	[5.474,5.590]	+
13.15566	-0.697930	0.93926	1/2	[5.474,5.590]	+
13.15748	-0.698026	0.95484	1/2	[5.534,5.591]	+
14.06146	-0.745984	0.83708	1/3	[5.156,5.393]	-
14.08066	-0.747002	0.85327	1/3	[5.114,5.347]	-
14.09948	-0.748001	0.86909	1/3	[5.098,5.332]	-
14.55725	-0.772286	0.88706	1/4	[5.253,5.574]	-
16.64808	-0.883208	1.19288	1/3	[3.830,3.964]	-
16.76479	-0.889400	1.37020	1/3	[6.487,6.492]	+
17.80747	-0.944715	2.06327	1/4	[3.189,3.517]	-
20.59152	-1.09242	1.45497	1/3	[6.276,6.278]	+

Varying masses

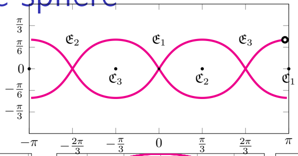


Varying masses

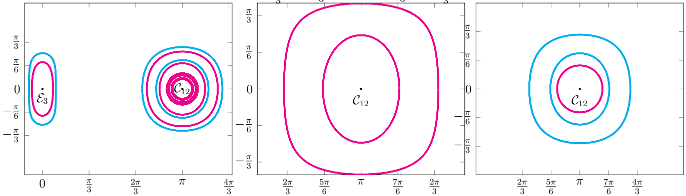
m_1	$m_1 + m_2 = 2$, $m_3 = 1.0$, $\omega = 1/2$				
	A	E	$\|J\|$	$[I_{\text{min}}, I_{\text{max}}]$	St
0.99	13.15748	-0.698026	0.95484	[5.534,5.591]	+
0.95	13.15312	-0.697795	1.01964	[5.470,5.588]	+
0.9	13.12580	-0.696348	1.09648	[5.456,5.575]	+
0.8	12.99779	-0.689554	1.23654	[4.722,5.568]	+
0.7	12.77091	-0.677518	1.35872	[4.054,5.800]	+

Periodic orbits on shape sphere

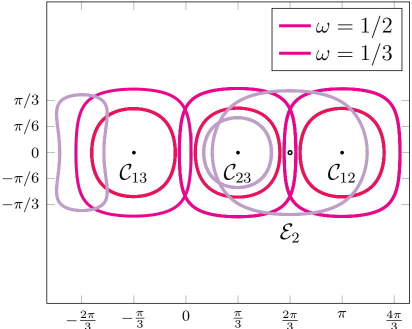
Figure-eight:



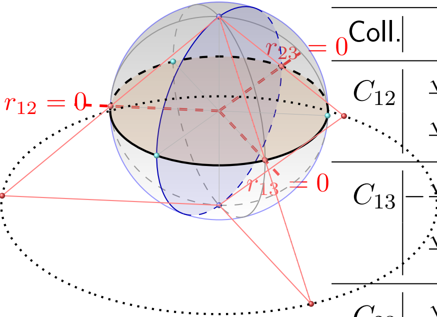
$2-1$:



Line:



Global regularization by Lemaitre



$$\zeta = z\frac{\sqrt{8} + z^3}{1 - \sqrt{8}z^3},$$

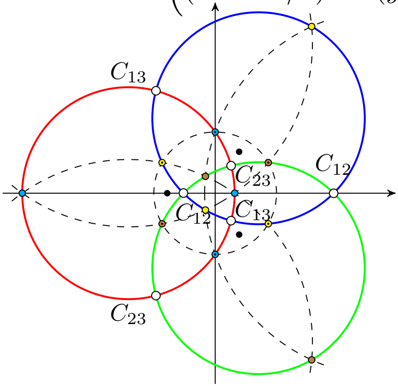
Coll.	extended complex plane z	ζ
C_{12}	$\frac{\sqrt{2}-\sqrt{6}}{2}$ $\frac{\sqrt{2}+\sqrt{6}}{2}$	-1
C_{13}	$-\frac{\sqrt{2}}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{6}}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\imath$ $\frac{\sqrt{2}}{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{6}}{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\imath$	$\frac{1}{2}-\frac{\sqrt{3}}{2}\imath$
C_{23}	$-\frac{\sqrt{2}}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{6}}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\imath$ $\frac{\sqrt{2}}{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{6}}{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\imath$	$\frac{1}{2}+\frac{\sqrt{3}}{2}\imath$

Preimages of unit circle

$$|\zeta|^2 = \zeta\bar{\zeta} = 1.$$

Direct calculations lead to the equation

$$\Big((x+\sqrt{2})^2 + y^2 - 3\Big)\Big((x-\sqrt{2}/2)^2 + (y-\sqrt{6}/2)^2 - 3\Big)\Big((x-\sqrt{2}/2)^2 + (y+\sqrt{6}/2)^2 - 3\Big) = 0$$



Degenetate trajectories; rectilinear orbits ($m_1 = m_2 = m_3 = 1$)

$$T = \frac{\dot{\xi_1}^2 + \dot{\xi_2}^2}{8\sqrt{\xi_1^2 + \xi_2^2}}$$

Polar coordinates of (ξ_1, ξ_2)

$$\begin{aligned} \xi_1 &= \varrho^2 \cos(\varphi), \\ \xi_2 &= \varrho^2 \sin(\varphi), \end{aligned}$$

$$T = \frac{1}{2}\left(\dot{\varrho}^2 + \frac{1}{4}\varrho^2\dot{\varphi}^2\right).$$

$$\begin{aligned} V &= 1/r_{12} + 1/r_{13} + 1/r_{23} = \\ &= \frac{1}{\varrho}\left(\frac{1}{\sqrt{1+\cos\varphi}} + \frac{1}{\sqrt{1-\cos(\varphi-\pi/3)}} + \frac{1}{\sqrt{1-\cos(\varphi-5\pi/3)}}\right) \\ &= \frac{1}{\varrho}\frac{1+4\cos\varphi}{\sqrt{1+\cos\varphi}(2\cos\varphi-1)} = \frac{D(\theta)}{\rho}. \end{aligned}$$

Singular values of φ: $\varphi = \pm\pi/3, \pi$.

Rectilinear trajectories

Energy integral

$$2h = \dot{\varrho}^2 + \frac{1}{4}\varrho^2\dot{\varphi}^2 - 2V = \dot{\varrho}^2 + \frac{1}{4}\varrho^2\dot{\varphi}^2 - 2\frac{D(\varphi)}{\varrho}\,. \tag{1}$$

Possible motion region (in coordinates ϱ, φ):

$$\frac{D(\varphi)}{\varrho} = \frac{1}{\varrho}\frac{1 + 4\cos\varphi}{\sqrt{1+\cos\varphi}(2\cos\varphi - 1)} \geq -h.$$

Zero velocity curve ($h = -1/2$) is given by

$$\varrho = \frac{2(1 + 4\cos\theta)}{\sqrt{1+\cos\theta}(2\cos\theta - 1)}\,. \tag{2}$$

Parametrization

$$-\sqrt{2}+\sqrt{3}\cos\psi+i\,\sqrt{3}\sin\psi \quad \text{vs} \quad z_E=\frac{\cos E}{\sqrt{3}+\sqrt{2}\cos E}+i\,\frac{\sqrt{3}\sin E}{\sqrt{3}+\sqrt{2}\cos E}$$

$$\begin{aligned}\cos\varphi &= \frac{5+\cos 4E}{7-\cos 4E},\\ \sin\varphi &= \frac{4\sqrt{3}\sin 2E}{7-\cos 4E}.\end{aligned}$$

then

$$V=\frac{1}{r_{12}}+\frac{1}{r_{13}}+\frac{1}{r_{23}}=\frac{1}{\varrho}\,\frac{\sqrt{7-\cos 4E}\,(9+\cos 4E)}{2\sqrt{3}(1+\cos 4E)}=\frac{D(E)}{\varrho}\,,$$

Hamiltonian

$$H=1/2\left(p_\varrho^2+\frac{(7-\cos 4E)^2p_E^2}{24\varrho^2(1+\cos 4E)}\right)-\frac{1}{\varrho}\,\frac{\sqrt{7-\cos 4E}\,(9+\cos 4E)}{2\sqrt{3}(1+\cos 4E)}$$

Parametrization

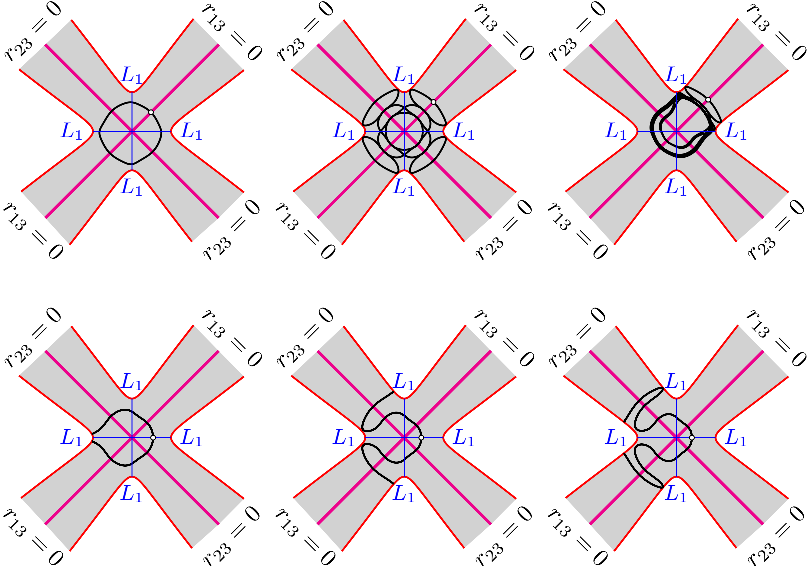
Not singular Hamiltonian

$$H' = (1 + \cos 4E)(H - h) =$$
$$= \frac{1}{2}(1 + \cos 4E)(p_\rho^2 - 2h) + \frac{(7 - \cos 4E)^2 p_E^2}{48\rho^2} - \frac{\sqrt{7 - \cos 4E}(9 + \cos 4E)}{2\sqrt{3}\rho},$$

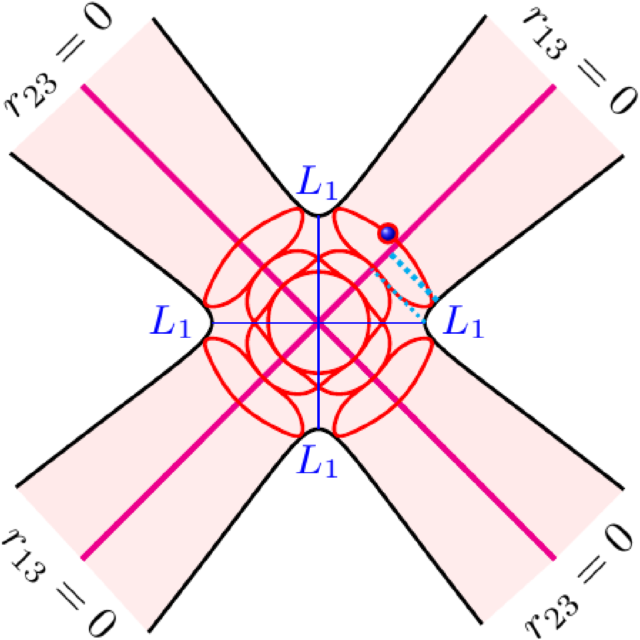
On the collision ray $E = \pi/4 + k\pi/2$:

$$p_E^2 = 2\sqrt{6}\varrho.$$

Rectilinear trajectories, results



$\rho = 3.74531687$



Properties of rectilinear trajectories

- $\dot{E} = 0, E = 0 \Rightarrow$ homothetic motion
- All other trajectories intersect the rays $E = \pi/4 + k\pi/2$ и $E = k\pi/2$
- Free-fall trajectories are orthogonal to the zero velocity curve
- Rectilinear trajectories are orthogonal to collision rays
- The equality $p_E^2 = 2\sqrt{6}\rho$ holds on the collision rays $E = \pi/4 + k\pi/2$
- The maximum of ϱ is achieved together with the maximum of p_E
- The intersection with Eulerian lines occurs between a series of collisions
- $\lim\limits_{\rho\to\infty} l_\theta = \sqrt{2}$; $\lim\limits_{\rho\to\infty} l_E \sim \sqrt{\rho}$.
- $\inf l_\theta$ is achived when $\rho \to \infty$
  $\min l_E$ is achived when $E = \pi/4 + k\pi/2 \pm 0.605632$ and is equal to $l_E \approx 2.285948$.

Chaos

- Each orbit intersects the ray $E = \pi/4$ and $p_{E_0}^2 = 2\sqrt{6}\rho_0$, the set of all orbits are defined by the number $\rho_0 \in \mathbb{R}^+$
- If initial point is located on the ray $E = 0$, then initial conditions are located in the region

$$\mathfrak{E} : 0 < \rho \leq \sqrt{5}/\sqrt{2},\ p_{E_0}^2 \leq \frac{4\rho(5\sqrt{2} - 2\rho)}{3}.$$

The mapping $\mathfrak{T} : (0, \infty) \to \mathfrak{E}$ translates one-dimensional line to two-dimensional set $\mathfrak{E}$ (similar to Peano curve?)

Isosceles trajectories

Parametrization:

$$z_E = \frac{\sqrt{2}\cos E}{\sqrt{3}-\cos E} + i\,\frac{\sqrt{3}\sin E}{\sqrt{3}-\cos E}\,.$$

Coordinates

$$\begin{aligned}
\cos\theta &= \frac{(\cos 2E-5)(1+3\cos 2E)}{3\cos^2 2E+2\cos 2E+11}\,,\\
\sin\theta &= \frac{-8\sqrt{3}\sin E\sin 2E}{3\cos^2 2E+2\cos 2E+11}\,,
\end{aligned}$$

Hamiltonian

$$\begin{aligned}
H = 1/2\left(p_\varrho^2 + \frac{(3\cos^2 2E+2\cos 2E+11)^2 p_E^2}{24\varrho^2(1-\cos 2E)(3+\cos 2E)^2}\right)\\
-\frac{1}{\varrho}\,\frac{\sqrt{3\cos^2 2E+2\cos 2E+11}\,(7-3\cos 2E)}{\sqrt{6}(1-\cos 2E)(3+\cos 2E)}\,,
\end{aligned}$$

Isosceles trajectories, results

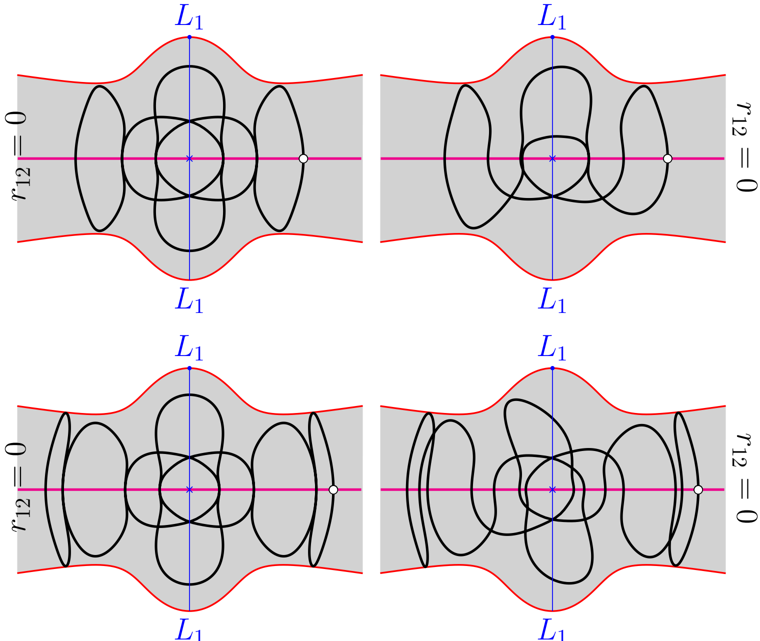
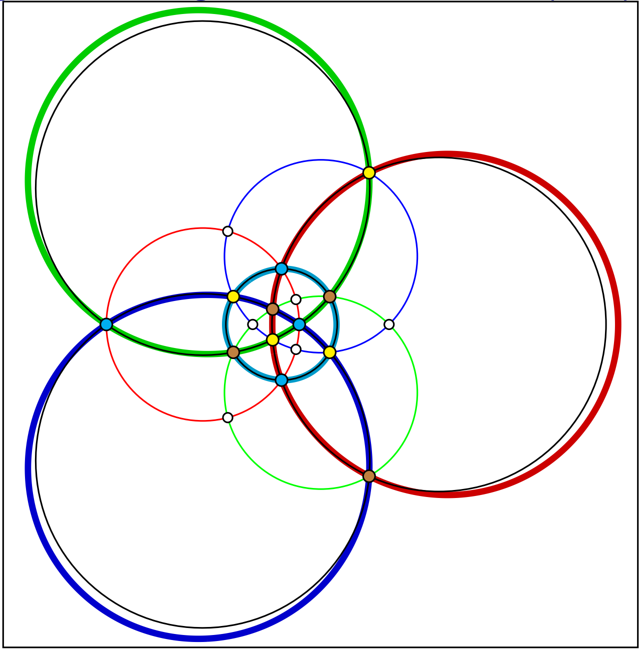
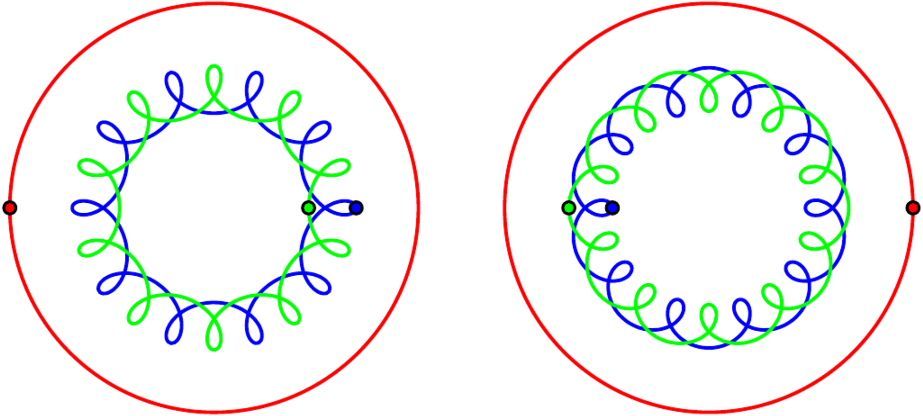


Figure-eight on the regularized extended complex plane





THANKS!!!