General three body problem in the shape space

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Abstract. The general three-body problem is considered in the shape space, the space reduced by translation and rotation. In this space the symmetric periodic orbits are studied, as well as the degenerate (collinear and isosceles) orbits. The Lemaitre regularization are used to regularize the collisions for degenerate orbits. The regions of possible motion for differen values of angular momentun are constructed in the shape space.

1. Introduction

The shape space is the quotient of $\mathbb{R}^n$ by translations, rotations and scaling. For N-body problem we have several ways to reduce configuration space by translations. We can use, for example, baricentric coordinates. The more convinient way is to use Jacobi coordinates:

$$\begin{aligned} \mathbf{Q}_1 &= \mathbf{r}_2 - \mathbf{r}_1, \\ \mathbf{Q}_2 &= \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \end{aligned}$$

So, for planar problem we have 4-dimensional configuration space.

The sphere in the space of Jacobi coordinates $(\mathbf{Q}_1, \mathbf{Q}_2)$ is $\mathbb{S}^3$, and, by eliminating rotations, we obtain $\mathbb{S}^2$. Thus, we naturally arrive at the classical Hopf transformation ($\mathcal{S}^1 \hookrightarrow \mathcal{S}^3 \to \mathcal{S}^2$):

$$\begin{aligned} \xi_1 &= \frac{1}{2}\mu_1|\mathbf{Q}_1|^2 - \frac{1}{2}\mu_2|\mathbf{Q}_2|^2, \\ \xi_2 + i\xi_3 &= \sqrt{\mu_1\mu_2}\,\mathbf{Q}_1\bar{\mathbf{Q}}_2. \end{aligned} \tag{1}$$

2. Periodic Orbits

In [1] it is shown that the symmetry groups of the general planar three-body problem are exhausted by 10 groups. Three of these groups served as the basis for the search for symmetric periodic solutions [2]. The found trajectories can

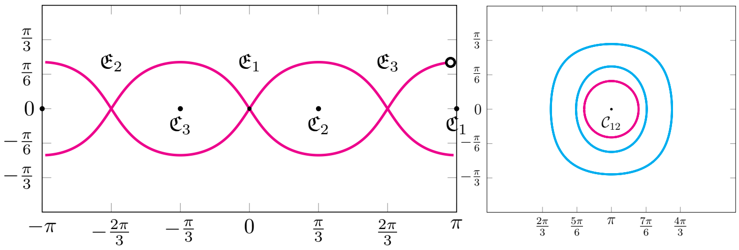

FIGURE 1. Figure-eight (left) and $2-1$-symmetry orbits (right) on shape sphere

be mapped into the shape space, such a transformation is unambiguous up to the rotation of the original barycentric system. Since the distance from the origin varies little in the shape space for these trajectories (within $\pm 10\%$), then for qualitative analysis we can limit ourselves to their projection onto the sphere of shapes.

Three symmetries from the list of planar three-body problem symmetries are studied: simple choreography (only one orbit – the figure-eight), 2-1 choreographies (where two masses must be equal), and linear symmetry (where all masses differ from each other). The obtained solutions are analyzed.

Some of symmetrical orbits are shown on fig. 1.

3. The Regions of Possible Motion

If the energy constant is negative (for example, $h = -1/2$), there are five topologically distinct regions of possible motion, depending on the value of the angular momentum constant J. The type of region changes when J reaches values corresponding to the Lagrange points $L_{4,5}$, L_3, L_2, and L_1. The situation is analogous to well-known surfaces of zero-velocity or Hill's surfaces in Restricted Three-Body Problem. Indeed, in the case for general three-body problem we have five topological type of surfaces depending on the value of angular momentum. It should be noted that the zero velocity surfaces in the circular restricted three-body problem are constructed in a rotating coordinate system, while in our case, they are in the shape space[3].

4. Regularization and Degenerate Orbits

For degenerate cases (collinear and isosceles trajectories) one need eliminate the singularity. For shape space the Lemaitre regularization is convinient enough. For the orbits under consideration a parameterization is constructed that allows the equations of motion for these degenerate cases, free from singularities. A lot of such orbits have been obtained numerically.

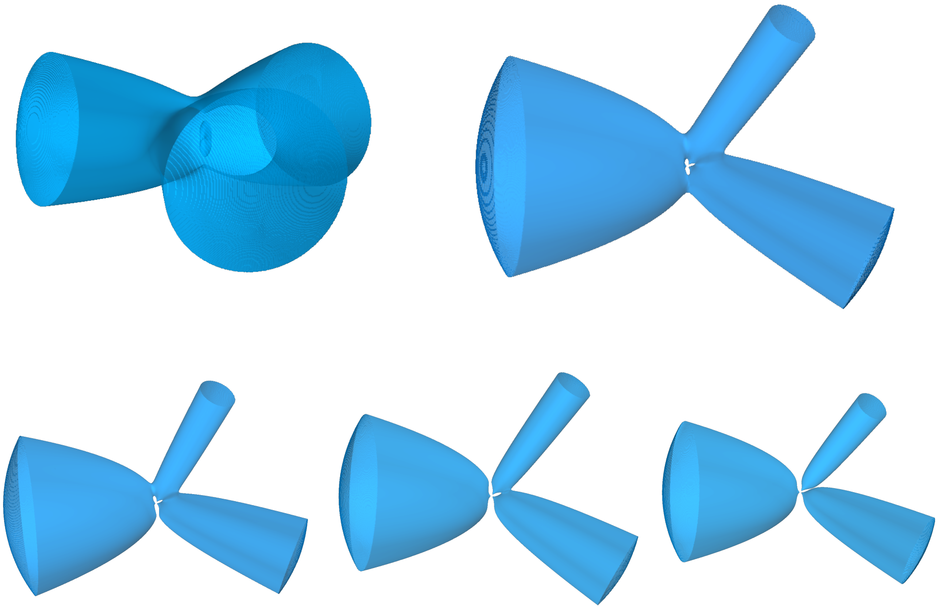

FIGURE 2. Regions of possible motion

The preimages of figure-eight orbit in configuretion space regularized by Lemaitre space are very closed to circles.

5. Conclusion

The three-dimensionality of this space makes it possible to simplify the analysis of solutions, and at least simply visualize the space of solutions.

The study of periodic orbits in the shape space allows us to conclude that at least some of them have a simple form on the shape sphere: the trajectories (topologically) are a circle in the center of which lies either a singular point $\mathcal{C}_i$, or an Eulerian point $\mathcal{E}_i$.

The shape space makes it possible to construct easy the zero-velocity surfaces, and, therefore, areas of possible motion.

The Lemaitre transform allows you to simply regularize degenerate orbits. Using the given parameterizations, we regularize the Hamiltonian, and solve the equations of motion, which have no singularities, numerically. At the same time, the properties of the solutions allow us to conclude that the motion is chaotic. The corresponding orbits are given for collinear and isosceles configurations.

References

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[3] Titov V., *The regions of possible motion of the general planar three-body problem (in russian).* Zap. nauchn. sem. POMI: **517**, 225–249. 2022.

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