APPROXIMATE THEORY OF A GYROSCOPE AND ITS APPLICATIONS TO THE MOTION OF SPACE OBJECTS

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Abstract. The motion of an axisymmetric rigid body with a fixed point under the action of a periodic torque is considered. Two small parameters are introduced: the first characterizes the smallness of the amplitude of the torque, and the second characterizes the smallness of the component of the kinetic moment perpendicular to the axis of symmetry. The smallness of the second small parameter is usually the basis for using the approximate theory of the gyroscope. Using this approximation, one can quite simply find the precession velocity of the top under the action of a small periodic torque. It is shown that the relative accuracy of the velocity calculated in this way is practically independent of the second small parameter, which does not exceed a value of the order of unity. In this way, a simple formula is found for the precession of the Earth's satellite under the action of the Earth's gravitational field. The resulting simple formula for the velocity of the Lunar-Solar precession of the Earth agrees well with astronomical observations.

Introduction

The motion of an axisymmetric rigid body is described by an equation for a unit vector $\mathbf{e}$ lying on the axis of symmetry [1]. The exact equation includes the second derivatives of the vector $\mathbf{e}$ with respect to time. In the case of rapid rotation, the approximate theory of a gyroscope proposes to ignore them. Then there remains a first-order equation with respect to the vector $\mathbf{e}$, which is called the equation of the precession theory of a gyroscope. From this equation, the precession velocity under the action of a periodic torque is easily found by the averaging method [2]. It is shown that the relative accuracy of the precession velocity is proportional to the amplitude of the torque and does not significantly depend on the component of the kinetic moment perpendicular to the axis of the top.

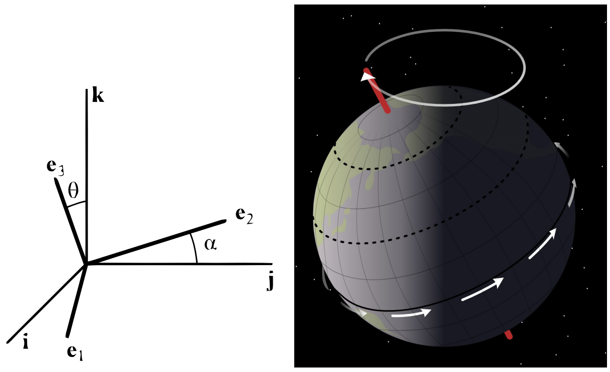

FIGURE 1. Angles of precession and nutation. Lunisolar precession.

1. Exact equations

The motion of an axisymmetric body with a fixed point lying on the axis of symmetry is conveniently described using a unit vector $\mathbf{e} = \mathbf{e_3}$ lying on the axis of symmetry. In this case, information about the rotation of the body about the axis of symmetry will not interest us. The equation for the vector can be obtained from the law of change of the kinetic moment [1]

$$\begin{gathered}\frac{d\mathbf{K}}{dt} = \mathbf{Mom}, \quad \mathbf{K} = A\mathbf{e} \times \frac{d\mathbf{e}}{dt} + Cr\mathbf{e} \\ \frac{d\mathbf{K}}{dt} = A\mathbf{e} \times \frac{d^2\mathbf{e}}{dt^2} + Cr\frac{d\mathbf{e}}{dt} + C\mathbf{e}\frac{dr}{dt} = \mathbf{Mom}\end{gathered} \tag{1}$$

where $\mathbf{K}$ is the kinetic moment, $\mathbf{Mom}$ is the moment of force applied to a point on the axis of symmetry, $\mathbf{e}$ is a unit vector directed along the axis of symmetry, A, C are the moments of inertia of a rigid body, r is the projection of the angular velocity onto the axis of symmetry. It is assumed that the vector $\mathbf{Mom}$ is a periodic function of the argument $\tau = \omega t$, ω is the frequency.

Let us introduce two dimensionless parameters $\varepsilon = \frac{max|\mathbf{Mom}|}{Cr\omega}$, $\varepsilon_1 = \frac{A\omega}{Cr}$ and assume that the projection of the moment of forces $\mathbf{Mom}$ on the axis $\mathbf{e} = 0$.

Then the system will be reduced to the following dimensionless form for angles of precession α and nutation θ (fig.1)

$$\begin{gathered}-\varepsilon_1\left(\ddot{\alpha}\sin\theta + 2\dot{\theta}\dot{\alpha}\cos\theta\right) + \dot{\theta} = \varepsilon M_1(\theta, \alpha, \tau), \\ \varepsilon_1\left(\ddot{\theta} - \dot{\alpha}^2\sin\theta\cos\theta\right) + \dot{\alpha}\sin\theta = \varepsilon M_2(\theta, \alpha, \tau), \quad \tau = \omega t\end{gathered} \tag{2}$$

Here the dots denote the derivatives with respect to τ. The parameter ε_1 determines the ratio of the first terms on the left-hand side of the equations to the second. For $\varepsilon_1 << 1$, the approximate theory of the gyroscope is usually used, in which the first terms are discarded [1]

$$\frac{d\theta}{d\tau} = \varepsilon M_1(\theta, \alpha, \tau), \quad \frac{d\alpha}{d\tau} \sin\theta = \varepsilon M_2(\theta, \alpha, \tau) \tag{3}$$

For the components of the moment with a small parameter ε, the (3) system can be easily studied by the averaging method.

It seems obvious that the relative error of the approximate theory of the gyroscope (3) is proportional to the parameter ε_1. However, this is not so. It is shown that the relative error of the nutation and precession angles determined by the approximate theory of the gyroscope (3) is proportional to the parameter ε for almost all values of the parameter ε_1 limited by a number of the order of unity. The importance of this statement follows from the fact that there are many problems in mechanics in which the parameter ε_1 significantly exceeds the parameter ε in magnitude.

2. Formulation of the theorem and examples of its application

Theorem For the complete system of equations (2) with 2π periodic in τ components of the moment of force M_i with small parameters ε and ε_1, the precession angle is determined from the system of equations (3) with a relative error of the order of ε and for almost any small values of ε_1 is approximated by the averaged system (3).

Example 1. Precession of a body in the two-body problem. Consider the circular two-body problem, in which the first body is a rigid body of mass m, and the second has mass M. The bodies are attracted by the law $\mathbf{F} = -\gamma M m \frac{\mathbf{r}}{|\mathbf{r}|}$.

A body of mass m moves under the action of force $\mathbf{F}$ along a circle of radius R_1, the center of which is located at the center of mass of the bodies. Due to the inhomogeneity of the field, a moment of force acts on a solid body of mass relative to its center of mass

$$\mathbf{Mom}(\omega t) = \frac{3\gamma M m}{R^3}(A - C)\tilde{\mathbf{M}}, \quad \tilde{\mathbf{M}} = ((\mathbf{r}_0/R) \cdot \mathbf{e})((\mathbf{r}_0/R) \times \mathbf{e}) \tag{4}$$

where $\mathbf{r}_0$ is the radius vector from the center of the body of mass m to the center of the body M , A and B are the moments of inertia of the body relative to the axis of symmetry and the axis perpendicular to it $R = |\mathbf{r}_0|$.

The bodies move in circular orbits relative to the center of mass and the distance between the bodies remains constant. The circular orbit is in the plane of vectors **i**, **j**.

According to the theorem, it is sufficient to solve the simplified system of equations

$$\frac{d\theta}{d\tau} = \varepsilon M_1(\theta, \alpha, \tau), \quad Cr\frac{d\alpha}{d\tau} \sin\theta = \varepsilon M_2(\theta, \alpha, \tau), \quad \frac{dr}{d\tau} = 0$$

$$M_1 = \sin\theta \cos(\tau - \alpha)\sin(\tau - \alpha), \quad M_2 = -\sin\theta\cos\theta\cos^2(\tau - \alpha).$$

By averaging the right-hand sides, we obtain

$$\dot{\alpha} = -\frac{1}{2}\varepsilon\cos\theta, \quad \dot{\theta} = 0, \quad \varepsilon = \frac{3\gamma M}{R^3 r\omega}\delta == 3\frac{\omega}{r}\left(1 + \frac{m}{M}\right)^{-1}$$

From here we obtain the formula for the angular velocity of precession

$$\frac{d\alpha}{rdt} = -\frac{3}{2}\left(\frac{\omega}{r}\right)^2 \delta\cos\theta\left(1 + \frac{m}{M}\right)^{-1}$$

Example 2. Lunisolar precession (fig 1). It consists of the angular velocities of the solar and lunar precessions

$$\frac{d\alpha_1}{dt} = \frac{3}{2}\frac{\omega_1^2}{r}\delta\cos\theta_1, \quad \frac{d\alpha_2}{dt} = \frac{3}{2}\frac{\omega_2^2}{r}\delta\cos\theta_2\,(1 + m/M)^{-1}$$

The angle of inclination of the plane of the Earth's equator to the plane of the Earth's rotation around the Sun varies periodically between the values $22.5° < \theta < 24.5°$ [4, 5]. The angle of inclination of the plane of the Moon's rotation around the Earth to the plane of the Earth's rotation around the Sun varies within the range of $5° < \phi < 5.28°$.

Following Beletsky, we accept the following average values

$$\theta_1 = \theta_2 = 23.5°, \quad \phi = 0°, \quad \omega_1 = (360/N_1)\circ/day, \quad \omega_2 = (360/N_2)\circ/day,$$
$$r = 360°/day, \quad N_1 = 365day, \quad N_2 = 28day, \quad \delta = 0.0033$$

The ratio of the masses of the Earth and the Moon is $m/M = 81$, and the ratio of the masses of the Earth and the Sun is neglected. Substituting these data for the velocity and period of precession, we get

$$\frac{d\alpha}{dt} == \frac{3}{2}\delta\cos\theta\left(\frac{\omega_1^2}{r} + \frac{\omega_2^2}{r}(1 + m/M)^{-1}\right)\frac{\circ}{day}, \quad P = \frac{360}{N_1 d\alpha/dt} = 26171year$$

Modern observations give a close value of $P = 25772year$.

Remark. After averaging the force function over the precession angle, and then over the true anomaly for the precession period, Beletski obtained a formula for the precession period that was similar in structure ([3] p. 209), but it was apparently given with typos.

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