On Algorithms of Hamiltonian Normal Form

Alexander Petrov and Alexander Batkhin

Abstract. The method of invariant normalization proposed by V.F. Zhuravlev, which is used for autonomous Hamiltonians for normal or symmetrized forms, is discussed. Normalizing canonical transformation is represented by a Lie series using a generating Hamiltonian. This method has a generalization proposed by A.G. Petrov, which normalize not only autonomous but also non-autonomous Hamiltonians. Normalizing canonical transformation is represented by a series using a parametric function. For autonomous Hamiltonian systems, the first two steps of approximations of both methods coincide, while the remaining steps differ. The normal forms in both methods coincide.

A method for testing normalization software is proposed. For this purpose the Hamiltonian of a strongly nonlinear Hamiltonian system is found for which the normal form is a quadratic Hamiltonian. The normalizing transformation is expressed in elementary functions.

1. Algorithm of invariant normalization

Normalization using Lie series is implemented as follows (see [1, 2, 3]). Let $H(\mathbf{q},\mathbf{p}) = H_0(\mathbf{q},\mathbf{p}) + F$ be the initial Hamiltonian, H_0 the principal term and F the perturbation. Its normal form (NF) $h(\mathbf{Q},\mathbf{P})$ and the generator of the Lie substitution $G(\mathbf{Q},\mathbf{P})$ are searched in the form of series

$$\begin{gathered} H(\mathbf{q},\mathbf{p}) = H_0(\mathbf{q},\mathbf{p}) + F, \quad h(\mathbf{Q},\mathbf{P}) = H_0(\mathbf{Q},\mathbf{P}) + f, \\ \%[2ex].F = \sum_{k=1}^{\infty} \varepsilon^k F_k(\mathbf{q},\mathbf{p}), \quad f = \sum_{k=1}^{\infty} \varepsilon^k f_k(\mathbf{Q},\mathbf{P}), \quad G = \sum_{k=1}^{\infty} \varepsilon^k G_k(\mathbf{Q},\mathbf{P}). \end{gathered} \tag{1}$$

Then for the NF $h = H_0 + f$ we get the Lie series

$$\begin{aligned} f =& H0 * G + M, \\ M =& F + F * G + \frac{1}{2!}(H_0 + F) * G^2 + \frac{1}{3!}(H_0 + F) * G^3 + \cdots , \end{aligned} \tag{2}$$

where $*$ denotes the Poisson bracket and the expression $Q * G^n - n$- times Poisson bracket: $Q * G^n = Q * G^{n-1} * G$.

Hence for the coefficients of the series (1) on powers ε of the NF f_k and the generator G_k we obtain a chain of homological equations:

$$H_0 * f_k = 0, \quad f_k = H_0 * G_k + M_k, \quad k = 1, 2, \ldots, \tag{3}$$

where the term M_k depends on the values of H_j, h_j, F_j, f_j, G_j, $j < k$ obtained in the previous steps. The structure of M_k for an arbitrary value of k is described in [4]. The solution of the equations (3) in Zhuravlev's method is found using quadrature

$$. \int_0^t m_k(\mathbf{Q}, \mathbf{P})\, dt = t f_k(\mathbf{Q}, \mathbf{P}) + G_k(\mathbf{Q}, \mathbf{P}) + g(t), \tag{4}$$

where the expression $m_j(t, \mathbf{Q}, \mathbf{P})$ is obtained from M_k by substituting solutions of the unperturbed system with Hamiltonian $H_0(\mathbf{q}, \mathbf{p})$. In the case of semi-simple eigenvalues λ_k of the unperturbed system, the integration of the quadrature (4) is replaced by substitution followed by simplification of exponents of the form $\exp(\lambda_k t)$.

The canonical transformation through the Lie generator is represented by Lie series

$$\mathbf{q} = \mathbf{Q} + \mathbf{Q} * G(\mathbf{Q}, \mathbf{P}) + \frac{1}{2!}\mathbf{Q} * G^2 + \cdots, \quad \mathbf{p} = \mathbf{P} + \mathbf{P} * G(\mathbf{Q}, \mathbf{P}) + \frac{1}{2!}\mathbf{P} * G^2 + \cdots. \tag{5}$$

2. Normalization Algorithm with Parametric Function

An alternative way of canonical transformation via the parametric function $\Psi(\mathbf{x}, \mathbf{y})$ [5, 3] has the form

$$\begin{cases} \mathbf{q} = \mathbf{x} - \frac{1}{2}\Psi_{\mathbf{y}}, \\ \mathbf{p} = \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{x}}, \end{cases} \quad \begin{cases} \mathbf{Q} = \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}}, \\ \mathbf{P} = \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}. \end{cases}$$

Eliminating the parameters $\mathbf{x}$ and $\mathbf{y}$, we can represent this transformation in the form of series

$$\mathbf{q} = \mathbf{Q} + \mathbf{Q} * \Psi(\mathbf{Q}, \mathbf{P}) + \frac{1}{2!}\mathbf{Q} * \Psi^2 + \cdots, \quad \mathbf{p} = \mathbf{P} + \mathbf{P} * \Psi(\mathbf{Q}, \mathbf{P}) + \frac{1}{2!}\mathbf{P} * \Psi^2 + \cdots,$$

which have three terms the same as (5) precisely by substituting $G \to \Psi$. The subsequent expansion coefficients at powers of Ψ^3 and higher will be different.

Instead of the equation (2), we get the following equation:

$$\begin{aligned} f &= H_0 * \Psi + M, \\ M &= F\left(\mathbf{x} - \frac{1}{2}\Psi_{\mathbf{y}}, \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{x}}\right) - f\left(\mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}}, \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}\right) + f(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Whence we obtain an analogous chain of homological equations. Moreover, for the first two approximations the equations differ only by replacing the coefficients G_1, G_2 by Ψ_1, Ψ_2.

The algorithm is similar to the Zhuravlev invariant normalization algorithm. Zhuravlev quadrature (4) is replaced by

$$\int_{t_0}^{t} m_k(\xi, t_0, \mathbf{Q}, \mathbf{P}) d\xi = (t - t_0) f_k(t_0, \mathbf{Q}, \mathbf{P}) + \Psi_k(t_0, \mathbf{Q}, \mathbf{P}) + g(t). \qquad (6)$$

In this quadrature, the main property of the NF is preserved: the perturbed part of the Hamiltonian system with Hamiltonian $f(t, \mathbf{Q}, \mathbf{P})$ is the integral of the unperturbed part with Hamiltonian $H_0(t, \mathbf{Q}, \mathbf{P})$. This allows us to find an analytical solution of the problem using the theorem for such a system: the general solution of the Hamiltonian equations with Hamiltonian $h = H_0 + f$ is obtained by substituting into the unperturbed solution the solution of the system with perturbed Hamiltonian $f(0, \mathbf{q}, \mathbf{p})$. This algorithm can be applied to non-autonomous systems.

Example The Mathieu equation $\ddot{x} + x(1 + 3\delta \cos 2t) = 0$ can be written in Hamiltonian form with Hamiltonian

$$H = H_0 + F, \quad H_0 = \frac{1}{2}\left(x^2 + u^2\right), \quad F = \delta \frac{3}{2} x^2 \cos 2t$$

Without using the theory of the Mathieu equation, we construct by normalization the asymptotic solution of the first approximation at $\delta \ll 1$.

1. Find the solution to the unperturbed system.

$$x = X \cos(t - t_0) + U \sin(t - t_0), \quad u = -X \sin(t - t_0) + U \cos(t - t_0) \qquad (7)$$

2. We define the function $m(t, t_0, Q, P)$ by substituting the solution (7) into the perturbed part of the Hamiltonian

$$m(t, t_0, X, U) = \delta \frac{3}{2} (X \cos(t - t_0) + U \sin(t - t_0))^2 \cos 2t$$

3. Compute the integral within (t_0, t) of the function $m(t', t_0, X, U)$. In this integral, we need to isolate the linear in time $f(t_0, X, U)$, the time-independent summand $\Psi(t_0, X, U)$, and the periodic in time $g(t)$, with period average equal to zero. From the integral (6) we find the functions f, Ψ, φ:

$$\begin{aligned} f =& - 3\delta \left(\cos(2t_0) \left(U^2 - X^2\right) + 2XU \sin(2t_0)\right)/8, \\ \Psi =& - 3\delta \left(\sin(2t_0) \left(5X^2 + 3U^2\right) - 2XU \cos(2t_0)\right)/32, \\ g(t) =& - \frac{3\delta}{32}\left(\left(U^2 - X^2\right) \sin(4t - 2t_0) + 2XU \cos(4t - 2t_0) - 4\left(X^2 + U^2\right) \sin 2t\right). \end{aligned}$$

The first function is a perturbation of the NF, the second term defines a substitution of the variables

$$\begin{aligned} x =& X - \Psi_U(t, X, U) = X + 3\delta(3U \sin 2t - X \cos 2t)/16, \\ u =& U + \Psi_X(t, X, U) = 3\delta(-5X \sin 2t + U \cos 2t)/16, \end{aligned}$$

which symmetrizes the Hamiltonian to small orders of δ^2.

$$h = H_0 + f, \; H_0 = \frac{1}{2}\left(X^2 + U^2\right), \; f = \frac{3\delta}{8}\left(\left(X^2 - U^2\right) \cos 2t - 2XU \sin 2t\right).$$

It is easy to see that the perturbed part of f is an integral of the unperturbed part of H_0. The general solution of the Hamilton equations with Hamiltonian h is obtained by substituting into the unperturbed solution

$$X = q\cos t + p\sin t, \quad U = -q\sin t + p\cos t$$

solutions with perturbed Hamiltonian $f(0,q,p) = 3\delta\left(-p^2+q^2\right)/8$

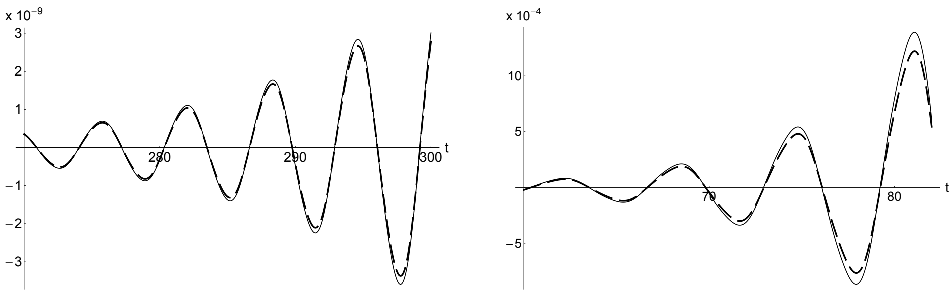
$$q = A\cosh\tau + B\sinh\tau, \quad p = A\sinh\tau - B\cosh\tau, \quad \tau = \frac{3}{4}\delta t,$$

Here is an example of constructing the asymptotic solution of the Mathieu equation with initial conditions $x(0)=1$, $\dot{x}(0)=0$

$$X = A\left(\cosh\tau\cos t - \sinh\tau\sin t\right), \quad U = -A\left(\cosh\tau\sin t - \sinh\tau\cos t\right),$$

$$x = X + \frac{3\delta}{16}(3U\sin 2t - X\cos 2t),$$

where $A = (1-(3/16)\delta)^{-1}$, $B=0$.



(a) $\delta = 0.1$. **(b)** $\delta = 0.2$.

FIGURE 1. Comparison of numerical and asymptotic solutions of the Mathieu equation.

In Fig. 1, the numerical solution of the Mathieu equation (solid line) is compared with the asymptotic solution (dashed line): (a) $\delta = 0.1$ and (b) $\delta = 0.2$. As can be seen, at $\delta = 0.1$ the asymptotic solution begins to differ slightly from the exact solution in the neighborhood of the maximum, when the function $x(t)$ reaches values of the order of 10^9. For larger values of $\delta = 0.2$ the difference becomes significant, when the function $x(t)$ reaches values of the order of 10^4.

3. Testing algorithms using tautochronous oscillations

To validate various normalization methods, it is useful to test them on nonlinear systems possessing tautochronous oscillations. For such systems, the NF in the neighborhood of the equilibrium position has the form of the Hamiltonian of a harmonic oscillator. Applying the invariant normalization method up to some fixed order all terms f_k of the NF should be zero.

An example of a tautochronous system is a system with Hamiltonian is given in [6]:

$$H = \frac{1}{2}\left(p^2 + (1+q)^2 + \frac{1}{(1+q)^2} - 2\right). \tag{8}$$

It can be shown that substituting the variables

$$q(Q,P) = \sqrt{R(Q,P)/2} - 1, \quad p(Q,P) = \frac{dq}{dt} = P\sqrt{\frac{P^2+4Q^2+4}{2R(Q,P)}}, \tag{9}$$

where $R(Q,P) = P^2 + 4Q^2 + 2 + 2Q\sqrt{P^2+4Q^2+4}$, has the following properties:

1. Differential form $PdQ - pdq$ is complete.
2. Substitution into the original Hamiltonian (8) converts it to the NF $h(Q,P) = (Q^2 + 4P^2)/2$.
3. Substitution the solution of the NF equations $Q = Q_0 \cos 2t$, $P = -2Q_0 \sin 2t$ into (9) gives the exact solution of the original Hamiltonian system.

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Alexander Petrov
Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia
e-mail: `petrovipmech@gmail.com`

Alexander Batkhin
Department of Aerospace Engineering, Techion – Israel Institute of Technology, Haifa, Israel
e-mail: `batkhin@technion.ac.il`