On Algorithms of Hamiltonian Normal Form

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Talk outlook

1. Introduction
2. Hamiltonian normal form
3. Algorithm of invariant normalization
4. Normalization algorithm with parametric function
5. Normalization algorithms testing by tautochronous oscillations

Abstract

The method of invariant normalization proposed by V.F. Zhuravlev, which is used for autonomous Hamiltonians for normal or symmetrized forms, is discussed. Normalizing canonical transformation is represented by a Lie series using a generating Hamiltonian. This method has a generalization proposed by A.G. Petrov, which normalize not only autonomous but also non-autonomous Hamiltonians. Normalizing canonical transformation is represented by a series using a parametric function. For autonomous Hamiltonian systems, the first two steps of approximations of both methods coincide, while the remaining steps differ. The normal forms in both methods coincide. A method for testing normalization software is proposed. For this purpose the Hamiltonian of a strongly nonlinear Hamiltonian system is found for which the normal form is a quadratic Hamiltonian. The normalizing transformation is expressed in elementary functions.

Introduction

Normal form (NF) of a system of ordinary differential equations (ODE) computed near an invariant manifold (stationary point, periodic solution or invariant torus) is an underrated powerful technique for investigation of local dynamics of the phase flow in the vicinity of this invariant structure.

The main applications of NF are studying of integrability, stability and bifurcations, searching first integrals and periodic solutions.

The special properties of Hamilton systems require specific algorithms for computation their NF. The goal of the presented work is to provide a procedure for constructing so called homological equation of any order, which is used in the modern normalization techniques

Remark on notations

- $|\mathbf{p}| = \sum_{j=1}^{n} |p_j|$ denotes vector norm
- For $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $\mathbf{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$ we denote the ***multi-index power*** by $\mathbf{x}^{\mathbf{p}} \equiv \prod_{j=1}^{n} x_j^{p_j}$ and ***scalar product*** by $\langle \mathbf{p}, \mathbf{x} \rangle \equiv \sum_{j=1}^{n} p_j x_j$
- For a pair of functions F, G we denote a ***Poisson bracket*** by $\{F, G\} \equiv F * G$: $\{F, G\} \equiv \langle J \operatorname{grad} G, \operatorname{grad} F \rangle$, where J is the symplectic unit
- Notation $F * G^n$, $n > 1$ means multiple left-associated Poisson bracket with the following recurrent definition: $F * G^n = F * G^{n-1} * G$

Hamiltonian at stationary point

We consider an analytic Hamiltonian system

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \quad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{1}$$

with n degrees of freedom near its stationary point $\mathbf{x} = \mathbf{y} = 0$.

The Hamiltonian function $H(\mathbf{x}, \mathbf{y})$ is expanded into convergent power series

$$H(\mathbf{x}, \mathbf{y}) = \sum H_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}}$$

with constant coefficients $H_{\mathbf{pq}}$, $\mathbf{p}, \mathbf{q} \geqslant 0$, $|\mathbf{p}| + |\mathbf{q}| \geqslant 2$.

Linear part of the Hamiltonian system

Canonical transformations of coordinates $\mathbf{x}, \mathbf{y}$

$$\mathbf{x} = \mathbf{f}(\mathbf{u}, \mathbf{v}), \quad \mathbf{y} = \mathbf{g}(\mathbf{u}, \mathbf{v}), \tag{2}$$

preserve the Hamiltonian character of the initial system (1).

Let $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}$ be a phase vector. The linear part of the system (1) has the form

$$\dot{\mathbf{z}} = B\mathbf{z}, \quad B = \frac{1}{2} J \,\mathrm{Hess}\, H\big|_{\mathbf{z}=0},$$

where J is a symplectic unit matrix and $\mathrm{Hess}\, H$ is a Hessian of H.

Eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ of B can be reordered in such a way that $\lambda_{j+n} = -\lambda_j$, $j = 1, \ldots, n$. Denote by $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ a basic frequencies vector.

Hamiltonian normal form

According to [Bruno, 1972, § 12, Theorem 12] there exists a canonical formal transformation (2) in the form of power series, which reduces the initial system (1) into its ***normal form***

$$\dot{\mathbf{u}} = \partial h/\partial \mathbf{v}, \quad \dot{\mathbf{v}} = -\partial h/\partial \mathbf{u}$$

defined by the normalized Hamiltonian $h(\mathbf{u}, \mathbf{v})$

$$h(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} \lambda_j u_j v_j + \sum h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}}$$

containing only resonant terms $h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}}$ with ***resonant equation*** $\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0$.
Here $0 \leqslant \mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$, $|\mathbf{p}| + |\mathbf{q}| \geqslant 2$ and $h_{\mathbf{pq}}$ are constant coefficients.

Methods of constructing canonical transformations

There exist several methods for constructing canonical transformations

- Method of Jacobi generating functions [Birkhoff, 1966; Gustavson, 1966];
- Method of Lie series implemented as Deprit-Hori [Deprit, 1969; Hori, 1966] or invariant normalization [Zhuravlev, 1997] methods;
- Method of Poincare parametric generating function [Petrov, 2004];
- Method of primitive functions [Haro, 2002].

Normalization by the method of Lie series (1)

Using ***generating Hamiltonian*** $G(\mathbf{X}, \mathbf{Y})$ one defines a canonical system

$$\frac{\partial \mathbf{X}}{\partial \tau} = \frac{\partial G}{\partial \mathbf{Y}}, \quad \frac{\partial \mathbf{Y}}{\partial \tau} = -\frac{\partial G}{\partial \mathbf{X}} \tag{3}$$

with initial conditions $\mathbf{X}(0) = \mathbf{x}$, $\mathbf{Y}(0) = \mathbf{y}$. Generating function G is presented in the form of a series of a small parameter ε:

$$G(\mathbf{X}, \mathbf{Y}, \varepsilon) = \varepsilon G_1 + \varepsilon^2 G_2 + \cdots .$$

Let for $\tau = 1$ new variables $\mathbf{u}, \mathbf{v}$ are defined as the solution to the Cauchy problem (3): $\mathbf{u} = \mathbf{X}(1), \mathbf{v} = \mathbf{Y}(1)$. Transformation $(\mathbf{x}, \mathbf{y}) \to (\mathbf{u}, \mathbf{v})$ on the phase flow of the Hamiltonian system (1) is an univalent canonical transformation.

Normalization by the method of Lie series (2)

For small values of ε old variables $\mathbf{x}, \mathbf{y}$ can be written as Lie series of the new ones $\mathbf{u}, \mathbf{v}$:

$$\begin{aligned} \mathbf{x} &= \mathbf{u} + \mathbf{u} * G + \frac{1}{2!}\mathbf{u} * G^2 + \frac{1}{3!}\mathbf{u} * G^3 + \cdots\,, \\ \mathbf{y} &= \mathbf{v} + \mathbf{v} * G + \frac{1}{2!}\mathbf{v} * G^2 + \frac{1}{3!}\mathbf{v} * G^3 + \cdots \end{aligned} \tag{4}$$

New Hamiltonian h is also related to the old one H by the following Lie series

$$h(\mathbf{u}, \mathbf{v}) = H(\mathbf{u}, \mathbf{v}) + H * G + \frac{1}{2!}H * G^2 + \frac{1}{3!}H * G^3 + \cdots$$

Complex normal from (1)

The procedure of nonlinear normalization of the real Hamiltonian H is more convenient to perform, when it is written in complex form in variables $\mathbf{z}, \bar{\mathbf{z}}$.

There is a formal transformation $\Phi : (\mathbf{x}, \mathbf{y}) \to (\mathbf{z}, \bar{\mathbf{z}})$ which reduces the original Hamiltonian H to the form

$$h(\mathbf{z}, \bar{\mathbf{z}}) = \sum h_{\mathbf{pq}} \mathbf{z}^{\mathbf{p}} \bar{\mathbf{z}}^{\mathbf{q}}, \tag{5}$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$ and $|\mathbf{p}| + |\mathbf{q}| \geqslant 2$.
The value of $|\mathbf{p}| + |\mathbf{q}|$ is called the ***order*** of the corresponding expansion term.

Complex normal from (2)

Definition 1.

The Hamiltonian function $h(\mathbf{z}, \bar{\mathbf{z}})$ is called a ***complex normal form*** for semi-simple case if

1. its quadratic part $h_{\mathbf{0}}$ has the form $h_{\mathbf{0}} = \sum_{j=1}^{n} \lambda_j z_j \bar{z}_j$,
2. the expansion (5) contains only the terms

$$h_{\mathbf{pq}} \mathbf{z}^{\mathbf{p}} \bar{\mathbf{z}}^{\mathbf{q}}, \tag{6}$$

which satisfy ***resonant equation***

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0$$

The terms (6) that have $\mathbf{p} = \mathbf{q}$ are called ***secular***, all others are called ***strong resonant***.

Outlook of the normalization procedure

Here we describe normalization procedure

- Given the real Hamiltonian $H(\mathbf{x}, \mathbf{y})$ is transformed in the complex form $H(\mathbf{z}, \bar{\mathbf{z}})$
- One of the method of normalization is applied to $H(\mathbf{z}, \bar{\mathbf{z}})$ up to the definite order and we get it NF $h(\mathbf{Z}, \bar{\mathbf{Z}})$, which contains only resonant terms
- Obtained complex NF $h(\mathbf{Z}, \bar{\mathbf{Z}})$ can be transformed into the real NF $h(\mathbf{X}, \mathbf{Y})$

Hamiltonian near stationary point (1)

Applying scaling of the phase variables $\mathbf{x} \to \varepsilon\mathbf{x}$, $\mathbf{y} \to \varepsilon\mathbf{y}$ and independent variable $t \to \varepsilon^2 t$ to its Hamiltonian H near the SP it takes the form of power series in ε

$$H(\mathbf{x},\mathbf{y}) = H_0 + F = H_0 + \sum_{j=1}^{\infty} \varepsilon^j H_j(\mathbf{x},\mathbf{y})$$

where H_j is a homogeneous form of order $j+2$:

$$H_j = \sum_{|\mathbf{p}|+|\mathbf{q}|=j+2} H_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}}$$

Hamiltonian near stationary point (2)

The NF h of H is presented as a power series

$$h(\mathbf{z},\bar{\mathbf{z}}) = h_0 + f = h_0 + \sum_{j=1}^{\infty} \varepsilon^j h_j(\mathbf{z},\bar{\mathbf{z}}), \text{ where } h_0 = \sum_{j=1} \lambda_j z_j \bar{z}_j$$

and homogeneous forms h_j, $j > 0$, contain only resonant terms $h_{\mathbf{pq}}\mathbf{z}^{\mathbf{p}}\bar{\mathbf{z}}^{\mathbf{q}}$, $|\mathbf{p}| + |\mathbf{q}| = j + 2$, such that

$$\langle \boldsymbol{\lambda}, \mathbf{p} - \mathbf{q} \rangle = 0$$

Homological equations (1)

Transformation from original Hamiltonian H to its NF h is provided by the Lie generator (generating function) $G = \sum_{j=1} \varepsilon^j G_j$:

$$h = H + H * G + \sum_{j=2}^{\infty} \frac{1}{j!} H * G^j$$

Lie generator G produces a near identical transformation, so we have $h_0 = H_0$ and then

$$f = h_0 * G + M, \quad M = F + \sum_{j=1} \frac{1}{j!} H * G^j \tag{7}$$

Homological equations (2)

Collecting terms in equation (7) with equal factors ε^j, $j = 0,1,2,\ldots$ one can rewrite it as the recurrent system of ***homological equations***

$$h_0 * f_j = 0, \quad f_j = h_0 * G_j + M_j, \quad j = 1,2,\ldots, \tag{8}$$

where the term M_j depends on H_k, F_k, G_k, $k < j$, obtained on the previous steps

For small values of j terms M_j are the following

$$\begin{aligned} M_1 =& F_1, \quad M_2 = F_2 + F_1 * G_1 + \frac{1}{2} H_0 * G_1^2, \\ M_3 =& F_3 + F_1 * G_2 + F_2 * G_1 + \frac{1}{2} H_0 * (G_1 * G_2 + G_2 * G_1) + \frac{1}{2} F_1 * G_1^2 + \frac{1}{6} H_0 * G_1^3 \end{aligned}$$

Homological equations (3)

The simplified structure of M_j for an arbitrary value of j is described in [Batkhin, 2023]:

$$M_j = F_j + \frac{1}{2}\sum_{k=1}^{j-1} f_k^+ * G_{j-k} + \sum_{k=1}^{[j/2]} \frac{B_{2k}}{(2k)!} \sum_{(i_1,\ldots,i_{2k+1})\in\mu^j_{2k+1}} f^-_{i_1} * G^{2k}_{i_2\cdots i_{2k+1}}.$$

Here B_{2k} are Bernoulli numbers, $f_j^+ \equiv F_j + f_j$, $f_j^- \equiv F_j - f_j$, $H * G^k_{j_1\cdots j_k} = H * G^{k-1}_{j_1\cdots j_{k-1}} * G_{j_k}$, the set μ^j_{2k+1} contains all the tuple of $2k+1$ indices which sum is equal to j.

Solution of homological equations

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- ***Algebraic method***, which was independently developed by G. Hori [Hori, 1966] and A. Deprit [Deprit, 1969] and improved in subsequent works (see [Giacaglia, 1972; Markeev, 1978]. Homological equations are solved as a system of linear algebraic equations of the coefficients of homogeneous forms f_j and G_j. This method has no restrictions on the structure of the quadratic part h_0 and can be applied in the case of multiple or zero eigenvalues as well. Nevertheless, because of the large number of monomials, the order of the corresponding systems grows rapidly.

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- ***Method of invariant normalization***, proposed by V.F. Zhuravlev [Zhuravlev, 1997; Zhuravlev (et al.), 2015]. This method can be considered as subsequent averaging of functions M_j along the unperturbed solutions $\mathbf{z}(t, \mathbf{Z}, \bar{\mathbf{Z}})$. It can be applied for the case of nonzero simple eigenvalues.

Method of invariant normalization (1)

Along the solutions $\mathbf{z}(t, \mathbf{Z}, \bar{\mathbf{Z}})$, $\bar{\mathbf{z}}(t, \mathbf{Z}, \bar{\mathbf{Z}})$ to the unperturbed canonical system

$$\dot{\mathbf{z}} = \frac{\partial H_0}{\partial \bar{\mathbf{z}}}, \quad \dot{\bar{\mathbf{z}}} = \frac{\partial H_0}{\partial \mathbf{z}} \tag{9}$$

we have the following identities: $h_0 * f_j = df_j/dt = 0$ and $h_0 * G_j = dG_j/dt$.

So, homological equations can be rewritten in the form

$$\dot{f}_j = 0, \quad M_j = f_j - \dot{G}_j, \quad j = 1{,}2, \ldots$$

Method of invariant normalization (2)

After substitution the solutions $\mathbf{z}(t, \mathbf{Z}, \bar{\mathbf{Z}})$, $\bar{\mathbf{z}}(t, \mathbf{Z}, \bar{\mathbf{Z}})$ to the unperturbed system (9) into function M_j: $m_j(t, \mathbf{Z}, \bar{\mathbf{Z}}) = M_j(t, \mathbf{Z}, \bar{\mathbf{Z}})$ one gets the following quadrature

$$\int_0^t m_j(t, \mathbf{Z}, \bar{\mathbf{Z}})dt = t f_j(\mathbf{Z}, \bar{\mathbf{Z}}) + G_j(\mathbf{Z}, \bar{\mathbf{Z}}) + g(t) \qquad (10)$$

Hence, on each step of the normalization procedure the next term of the NF f_j is a coefficient at t, and the Lie generator term G_j is a time-independent term

Method of invariant normalization (3)

When all the eigenvalues $\boldsymbol{\lambda}$ are simple and purely imagine or real it is not necessary to integrate the left hand side of (10). All the functions M_j are homogeneous polynomials in $\mathbf{z}, \bar{\mathbf{z}}$, so after the substitution we get $m_j = \sum_k C_k \mathrm{e}^{\beta_k t} + C_0$, where β_k is a non-zero linear combination of the eigenvalues λ_j, $j = 1, \ldots, n$, and C_0, C_k are polynomials in new variables $\mathbf{Z}, \bar{\mathbf{Z}}$. It follows from (10) that

$$f_j = C_0, \quad G_j = \sum_k \frac{C_k}{\beta_k}$$

Remark

Method of invariant normalization can be applied to the systems with non-simple multiple eigenvalues by applying special scaling (see [Bruno, Petrov, 2006]).

Normalization algorithm with parametric function (1)

An alternative way of canonical transformation via the parametric function $\Psi(\boldsymbol{\xi}, \boldsymbol{\eta})$ [Petrov, 2004; Zhuravlev (et al.), 2015] has the form

$$\begin{cases} \mathbf{x} = & \boldsymbol{\xi} - \dfrac{1}{2}\Psi_{\boldsymbol{\eta}}, \\ \mathbf{y} = & \boldsymbol{\eta} + \dfrac{1}{2}\Psi_{\boldsymbol{\xi}}, \end{cases} \quad \begin{cases} \mathbf{u} = & \boldsymbol{\xi} + \dfrac{1}{2}\Psi_{\boldsymbol{\eta}}, \\ \mathbf{v} = & \boldsymbol{\eta} - \dfrac{1}{2}\Psi_{\boldsymbol{\xi}}. \end{cases}$$

Normalization algorithm with parametric function (2)

Eliminating the parameters $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, we can represent this transformation in the form of series

$$\mathbf{x} = \mathbf{u} + \mathbf{u} * \Psi(\mathbf{u}, \mathbf{v}) + \frac{1}{2!}\mathbf{u} * \Psi^2 + \cdots, \; \mathbf{y} = \mathbf{v} + \mathbf{v} * \Psi(\mathbf{u}, \mathbf{v}) + \frac{1}{2!}\mathbf{v} * \Psi^2 + \cdots,$$

which have three terms the same as (4) precisely by substituting $G \to \Psi$. The subsequent expansion coefficients at powers of Ψ^3 and higher will be different.

Normalization algorithm with parametric function (3)

Instead of the equation (7), we get the following equation:

$$\begin{aligned} f &= H_0 * \Psi + M, \\ M &= F\left(\mathbf{x} - \frac{1}{2}\Psi_{\boldsymbol{\eta}}, \boldsymbol{\eta} + \frac{1}{2}\Psi_{\boldsymbol{\xi}}\right) - f\left(\boldsymbol{\xi} + \frac{1}{2}\Psi_{\boldsymbol{\eta}}, \boldsymbol{\eta} - \frac{1}{2}\Psi_{\boldsymbol{\xi}}\right) + f(\boldsymbol{\xi}, \boldsymbol{\eta}). \end{aligned}$$

Whence we obtain an analogous chain of homological equations. Moreover, for the first two approximations the equations differ only by replacing the coefficients G_1, G_2 by Ψ_1, Ψ_2.

Normalization algorithm with parametric function (4)

The algorithm is similar to the Zhuravlev invariant normalization algorithm. Zhuravlev quadrature (10) is replaced by

$$\int_{t_0}^{t} m_k(\zeta, t_0, \mathbf{u}, \mathbf{v}) d\zeta = (t - t_0) f_k(t_0, \mathbf{u}, \mathbf{v}) + \Psi_k(t_0, \mathbf{u}, \mathbf{v}) + g(t). \tag{11}$$

In this quadrature, the main property of the NF is preserved: the perturbed part of the Hamiltonian system with Hamiltonian $f(t, \mathbf{u}, \mathbf{v})$ is the integral of the unperturbed part with Hamiltonian $H_0(t, \mathbf{u}, \mathbf{v})$. This allows us to find an analytical solution of the problem using the theorem for such a system: the general solution of the Hamiltonian equations with Hamiltonian $h = H_0 + f$ is obtained by substitution into the unperturbed solution the solution of the system with perturbed Hamiltonian $f(0, \mathbf{x}, \mathbf{y})$. This algorithm can be applied to non-autonomous systems.

Example: the Mathieu equation (1)

The Mathieu equation $\ddot{x} + x(1 + 3\delta\cos 2t) = 0$ can be written in Hamiltonian form

$$H = H_0 + F, \quad H_0 = \frac{1}{2}\left(x^2 + y^2\right), \quad F = \frac{3\delta}{2}x^2\cos 2t$$

Example: the Mathieu equation (2)

First approximation solution for $\delta \ll 1$

1. Solve the unperturbed system

$$x = u\cos(t-t_0) + v\sin(t-t_0), \quad y = -u\sin(t-t_0) + v\cos(t-t_0)$$

2. Define the function $m(t, t_0, Q, P)$

$$m(t, t_0, u, v) = 1.5\delta(u\cos(t-t_0) + v\sin(t-t_0))^2\cos 2t$$

3. From the integral (11) find the functions f, Ψ:

$$f = -\frac{3\delta}{8}\left(\left(v^2-u^2\right)\cos 2t_0 + 2uv\sin 2t_0\right), \Psi = -\frac{3\delta}{32}\left(\left(5u^2+3v^2\right)\sin(2t_0 - 2uv\cos 2t_0\right),$$

where f is a perturbation and Ψ defines variable transformation

Example: the Mathieu equation (3)

The general solution to the Hamilton equations with Hamiltonian h is obtained by substituting into the unperturbed solution

$$u = x\cos t + y\sin t, \quad v = -x\sin t + y\cos t$$

solutions with perturbed Hamiltonian $f(0, x, y) = \frac{3\delta}{8}\left(x^2 - y^2\right)$

$$x = A\cosh\tau + B\sinh\tau, \quad y = A\sinh\tau - B\cosh\tau, \quad \tau = \frac{3\delta}{4}t$$

Example: the Mathieu equation (4)

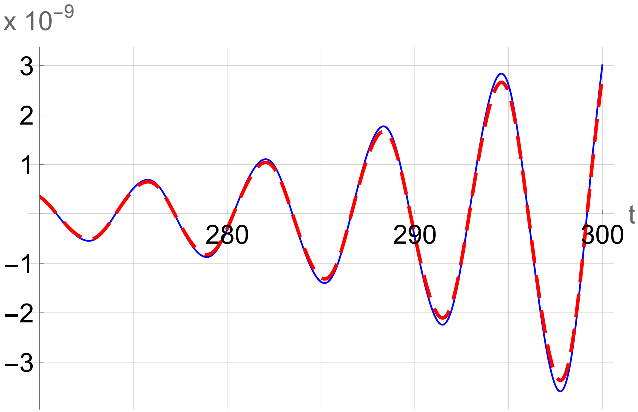

Figure 1: Comparison with exact solution for $\delta = 0.1$

Example: the Mathieu equation (5)

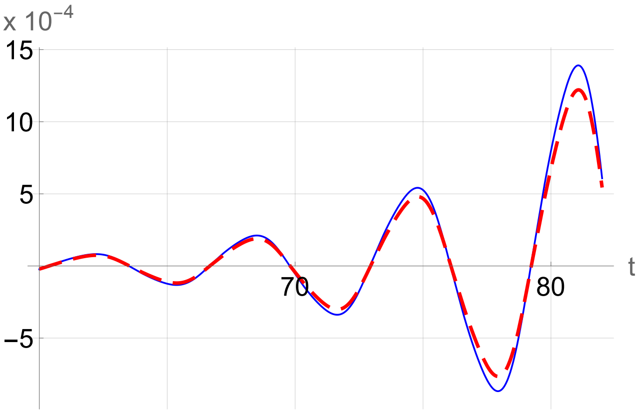

Figure 2: Comparison with exact solution for $\delta = 0.2$

Normalization algorithms testing by tautochronous oscillations (1)

To validate various normalization methods, it is useful to test them on nonlinear systems possessing tautochronous oscillations. For such systems, the NF in the neighborhood of the equilibrium position has the form of the Hamiltonian of a harmonic oscillator. Applying the invariant normalization method up to some fixed order all terms f_k of the NF should be zero.

An example of a tautochronous system is a system with Hamiltonian is given in [Petrov, 2024]:

$$H = \frac{1}{2}\left(p^2 + (1+q)^2 + \frac{1}{(1+q)^2} - 2\right). \tag{12}$$

Normalization algorithms testing by tautochronous oscillations (2)

It can be shown that substituting the variables

$$q(Q,P) = \sqrt{R(Q,P)/2} - 1, \quad p(Q,P) = \frac{dq}{dt} = P\sqrt{\frac{P^2+4Q^2+4}{2R(Q,P)}}, \tag{13}$$

where $R(Q,P) = P^2 + 4Q^2 + 2 + 2Q\sqrt{P^2+4Q^2+4}$, has the following properties:

1. Differential form $PdQ - pdq$ is complete.
2. Substitution into the original Hamiltonian (12) reduces it to the NF $h(Q,P) = \frac{1}{2}(P^2 + 4Q^2)$.
3. Substitution the solution of the NF equations $Q = Q_0 \cos 2t,\ P = -2Q_0 \sin 2t$ into (13) gives the exact solution of the original Hamiltonian system.

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Thanks for your attention!