On Types of Stability in Hamiltonian Systems

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Abstract. We consider conditions of three types of stability: Lyapunov, formal and weak of a stationary solution in a Hamiltonian system with a finite number of degrees of freedom. The conditions contain restrictions on the order of resonances and some inequalities for coefficients of the normal forms of the Hamiltonian functions. We also estimate the orders of solutions' divergence from the stationary ones under lack of formal stability.

1. Resonant normal form

Consider a Hamiltonian system

$$\dot{\xi}_j = \frac{\partial \gamma}{\partial \eta_j}, \quad \dot{\eta}_j = -\frac{\partial \gamma}{\partial \xi_j}, \quad j = 1, \ldots, n \tag{1}$$

with n degrees of freedom in the neighborhood of a stationary point at the origin

$$\boldsymbol{\zeta} \stackrel{\text{def}}{=} (\boldsymbol{\xi}, \boldsymbol{\eta}) = 0. \tag{2}$$

If the Hamilton function $\gamma(\boldsymbol{\zeta})$ is analytic at this point, then it expands into a convergent power series

$$\gamma(\boldsymbol{\zeta}) = \sum \gamma_{\mathbf{pq}} \boldsymbol{\xi}^{\mathbf{p}} \boldsymbol{\eta}^{\mathbf{q}}, \tag{3}$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$, $\mathbf{p}, \mathbf{q} \geqslant 0$, $\boldsymbol{\xi}^{\mathbf{p}} = \xi_1^{p_1} \cdots, \xi_n^{p_n}$, $\gamma_{\mathbf{pq}}$ are constant coefficients. Since the point (2) is stationary, the expansion (3) starts with quadratic terms. They correspond to the linear part of the system (1). The eigenvalues of its matrix are divided into pairs $\lambda_{j+n} = -\lambda_j$, $j = 1, \ldots, n$. Denote by vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ the set of *basic eigenvalues*. As known, canonical coordinate substitutions $\boldsymbol{\xi}, \boldsymbol{\eta} \to \mathbf{x}, \mathbf{y}$ preserve the Hamiltonian nature of the system.

Theorem 1 ([1, §12]). *There is a canonical formal transformation* $\boldsymbol{\xi}, \boldsymbol{\eta} \leftrightarrow \mathbf{x}, \mathbf{y}$ *that reduces the Hamiltonian* (3) *to the normal form*

$$g(\mathbf{x}, \mathbf{y}) = \sum g_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}}, \tag{4}$$

where the series g contains only resonant terms satisfying ***resonant equation*** $\langle \mathbf{p}-\mathbf{q}, \boldsymbol{\lambda}\rangle = 0$. *Here* $\langle\cdot,\cdot\rangle$ *means the scalar product.*

Condition A_k^n for system with n DOF takes place if the resonant equation has no integer solutions $\mathbf{p} \in \mathbb{Z}^n$ with $\|\mathbf{p}\| \leqslant k$.

This condition means that there are no resonances up to and including the order k. If it is satisfied, then in the normal form (4) is $g = \sum_{l=1}^{[k/2]} g_l(\boldsymbol{\rho}) + \tilde{g}^{(k)}(\mathbf{z}, \bar{\mathbf{z}})$, where $g_l(\boldsymbol{\rho})$ are homogeneous polynomials from $\rho_j = iz_j\bar{z}_j$, $j = 1, \ldots, n$, of degree l, and $\tilde{g}^{(k)}$ is a series from $\mathbf{z}, \bar{\mathbf{z}}$ starting with powers above k. In particular, under the condition A_2^n we have

$$g = \langle \boldsymbol{\rho}, \boldsymbol{\lambda}\rangle + \tilde{g}^{(3)}(\mathbf{z}, \bar{\mathbf{z}}),$$

and under the condition A_4^n we have

$$g = \langle \boldsymbol{\rho}, \boldsymbol{\lambda}\rangle + \langle C\boldsymbol{\rho}, \boldsymbol{\rho}\rangle + \tilde{g}^{(5)}(\mathbf{z}, \bar{\mathbf{z}}), \tag{5}$$

where C is $n \times n$ matrix.

2. Lyapunov and formal stabilities of stationary point

2.1. Lyapunov stability

Definition 1. A stationary point (SP) $\boldsymbol{\zeta} = 0$ of a real Hamiltonian system (1) is *stable by Lyapunov* if for every $\varepsilon > 0$ in “cube” $\|\boldsymbol{\zeta}\| < \varepsilon$ there exists a closed integral $(2n-1)$-dimensional manifold $\mathcal{L}$ surrounding the point $\boldsymbol{\zeta} = 0$ from all sides, where $\|\boldsymbol{\zeta}\| = \sum_{j=1}^{2n} |\zeta_j|$.

Lemma 1. *A SP* $\boldsymbol{\zeta} = 0$ *is Lyapunov stable if there exists a sign-definite real integral*

$$f(\boldsymbol{\zeta}) = f_l(\boldsymbol{\zeta}) + \tilde{f}^{(l)}(\boldsymbol{\zeta}) \tag{6}$$

of the system (1)*, where* $f_l(\boldsymbol{\zeta})$ *is a homogeneous form of degree* l*. In other words,*

$$\{f, \gamma\} = 0, \tag{7}$$

where $\{\cdot,\cdot\}$ *is the Poisson bracket, and* $f_l(\boldsymbol{\zeta})$ *does not equal to zero at any* $\boldsymbol{\zeta}$ *except the point* $\boldsymbol{\zeta} = 0$.

Stability is possible only if $\operatorname{Re} \boldsymbol{\lambda} = 0$.

Theorem 2 (Dirichlet). *Suppose* $\lambda_j = i\alpha_j$, $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, n$. *If the condition* A_2^n *is satisfied and the numbers* $\alpha_1, \ldots, \alpha_n$ *are of the same sign, then the SP* $\boldsymbol{\zeta} = 0$ *is stable according to Lyapunov.*

Here the role of the integral f is played by the Hamiltonian γ itself.

2.2. Formal stability

By *formal* we will mean power series, about the convergence of which nothing is known.

Definition 2 ([2]). A SP (2) of a real Hamiltonian system (1) is *formally stable* if there exists a formal real sign-defined integral (6) of the system (1), i.e., the formal identity (7) is satisfied and the homogeneous form f_l is null only at $\boldsymbol{\zeta} = 0$.

Formal stability means that the departure of solutions from the SP, if anything, is very slow: slower than any finite degree of t.

Definition 3 ([3, Ch. 4, § 4]). A SP (2) of a real Hamiltonian system (1) is *formally stable* if there exists a formal real integral

$$f(\boldsymbol{\zeta}) = f_l(\boldsymbol{\zeta}) + f_{l+1}(\boldsymbol{\zeta}) + \ldots + f_m(\boldsymbol{\zeta}) + \tilde{f}^{(m)}(\boldsymbol{\zeta})$$

of system (1), where $f_k(\boldsymbol{\zeta})$ are homogeneous forms of degree k and the sum

$$f^*(\boldsymbol{\zeta}) = f_l + f_{l+1} + \ldots + f_m \tag{8}$$

does not equal to zero in some neighborhood of the point $\boldsymbol{\zeta} = 0$ besides it.

Let $K \subset \mathbb{R}^n$ be a linear shell of integers $\mathbf{q}$ satisfying the equation $\langle \boldsymbol{\alpha}, \mathbf{q} \rangle = 0$, and $Q = \{\mathbf{q} \geqslant 0,\ \mathbf{q} \neq 0\} \subset \mathbb{R}^n$ is a non-negative orthant without origin.

Theorem 3 (Formal Stability Theorem [4]). *If Condition A_4^n is satisfied and in* (5)

$$\langle C\mathbf{q}, \mathbf{q} \rangle \neq 0 \text{ for } \mathbf{q} \in K \cap Q, \tag{9}$$

then the point $\boldsymbol{\zeta} = 0$ is formally stable in the sense of Definition 2

Here, the normal form of the Hamiltonian (4) from Theorem 1 is used to construct the formal integral.

In the situation when any resonance of multiplicity 1 takes place, there exists the only integral vector $k\mathbf{p}$, $k \in \mathbb{Z}\backslash\{0\}$, $\mathbf{p} \in \mathbb{Z}^n$, satisfying the resonant equation. Let $\boldsymbol{\omega}_j$, $j = 1, \ldots, n-1$, be the basis of the orthogonal complement to the one-dimensional solution space, then $\langle \boldsymbol{\omega}_j, \boldsymbol{\rho} \rangle$ is the first integral of the normalized system with Hamiltonian $g(\mathbf{z}, \bar{\mathbf{z}})$ [5, Ch. I, Sect. 3].

Lemma 2. *If there exists only one resonant vector $k\mathbf{p}$, $k \in \mathbb{Z}$, which does not belong to the positive orthant $\mathcal{Q}$, than SP $\boldsymbol{\zeta} = 0$ is formally stable.*

2.3. Method of formal stability investigation in a generic case with 3DOF

Consider a Hamiltonian system in the vicinity of the SP for which the following conditions are satisfied:

- the number of degrees of freedom of the system is greater than two;
- the quadratic form γ_2 in expansion (3) is nondegenerate and is not definite;
- the Hamiltonian function γ smoothly depends of the vector of parameters $\mathbf{P}$ from a domain $\Pi \subset \mathbb{R}^m$.

Corollary 1 (of Formal Stability Theorem 3). *If under the condition of Theorem 3 in $\mathbb{R}^3$ the intersection of the plane $\langle \boldsymbol{\lambda}, \mathbf{q}\rangle = 0$ and the cone $\langle C\mathbf{q}, \mathbf{q}\rangle$ either does not belong to $\mathcal{Q}$, or belongs to $\mathcal{Q} = \mathbb{R}^3_+$, but does not contain the integral vector $\mathbf{q}$, then the SP is formally stable.*

Definition 4. A *resonant variety* $\mathcal{R}^{\mathbf{p}}_n$ in the space K of coefficients $a_1, \dots, a_n$ of the semi-characteristic polynomial $\chi_n(\mu)$ of degree n is an algebraic variety, on which the vector of basis eigenvalues $\boldsymbol{\lambda}$ is a nontrivial solution to the resonant equation $\langle \mathbf{p}, \boldsymbol{\lambda}\rangle = 0$ for a fixed integer vector $\mathbf{p}^* \in \mathbb{Z}^n \setminus \{0\}$. An analytical representation of the variety $\mathcal{R}^{\mathbf{p}*}_n$ in an implicit or parametric form is denoted by $R^{\mathbf{p}^*}_n$.

To examine the formal stability of a SP of a Hamiltonian system (1), we should [6]:

- find in the space of parameters Π the stability set Σ of the linear system;
- find such domains, in which the quadratic form $\gamma_2(\mathbf{z})$ is not sign definite;
- find parts S_k in these domains that do not contain strong resonances;
- normalize the Hamiltonian in each of these parts S_k up to order four, and
- apply Formal Stability Theorem 3.

To do this, it is sufficient to select a point in each S_k in the space of parameters and use one of the normalization algorithms for the Hamiltonian function. Since all eigenvalues λ_k $(k = 1, \dots, n)$ are simple at each interior point of S_k, the invariant normalization algorithm can be easily applied.

Remark. Most of presented above statements are applicable for stability of a periodic solution.

3. Scattering order of solution

Let the function $f(t)$ be defined at real $t \to -\infty$. It is said to have *order* $\delta = \delta(t)$ if $\delta = \inf \varepsilon$ such that $f(t)/(-t)^\varepsilon \to 0$ at $t \to -\infty$. If $\delta > 0$, then $f(t)$ is unbounded, if $\delta < 0$, then $f(t) \to 0$ at $t \to -\infty$. In the latter case $\delta(f) < 0$, the larger δ is, the slower $f(t)$ approaches zero.

Definition 5. Let the solution $\boldsymbol{\zeta}(t)$ of the Hamiltonian system (1) tends to a SP (2) at $t \to -\infty$. On this solution *order of scattering* $\Delta = \min\{\delta \|\boldsymbol{\zeta}\|\}$.

Definition 6. *The scattering order* $\widetilde{\Delta}$ *of solutions* of the system (1) from the SP (2) is the lower bound of the scatter order Δ over all solutions $\boldsymbol{\zeta}(t)$ that tend to the point (2) at $t \to -\infty$.

The smaller $\widetilde{\Delta} < 0$, the faster the solutions are scattered from the SP. At formal stability the order of scattering of solutions from the SP is zero. Let us estimate the order of scattering $\widetilde{\Delta}$ in the absence of formal stability. The cases $-10^{-10} < \widetilde{\Delta} < 0$ can be considered as *weak stable.*

Conjecture. *Let the condition* A_2^n *and* $\varkappa = \min \|\mathbf{p}+\mathbf{q}\| > 2$ *by integer solutions* $\mathbf{p} \geqslant 0$, $\mathbf{q} \geqslant 0$ *of equation* $\langle \boldsymbol{\alpha}, \mathbf{p}-\mathbf{q} \rangle = 0$ *be satisfied, then the order of scatter of the system solutions* (1) *from the SP* $\widetilde{\Delta} \geqslant (2-\varkappa)^{-1}$.

4. Conclusion

These results were published in [7] together with:

1. more details, with examples;
2. number-theoretical approach simplifying the proofs of formal stability;
3. computing of formal stability in a complicated case;
4. similar theory for a neighborhood of a periodic solution.

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