On Types of Stability in Hamiltonian System

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Talk outlook

1. Resonant normal form
2. Lyapunov stability
3. Formal stability
4. Scattering order of solution
5. Conclusion

Abstract

We consider conditions of three types of stability: Lyapunov, formal and weak of a stationary solution in a Hamiltonian system with a finite number of degrees of freedom. The conditions contain restrictions on the order of resonances and some inequalities for coefficients of the normal forms of the Hamiltonian functions. We also estimate the orders of solutions' divergence from the stationary ones under lack of formal stability.

Nowadays there are three types of definitions of stationary-point stability in a Hamiltonian system:

- Lyapunov stability,
- formal stability by Moser and by Markeev,
- weak stability.

In the talk we present these definitions for a stationary point and give conditions on the Hamiltonian function which guarantee them. Formal stability investigation method in a generic case with 3DOF is considered. In the absence of formal stability one can consider a weak stability in the situation when the order of scattering of solutions is small. Therefore the order of scattering of the solution from a stationary point in the absence of formal stability is estimated.

Note on notation

Vectors are indicated in bold type. By default, these are vectors of dimension n unless otherwise specified, i.e. $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{p} = (p_1, \dots, p_n)$, and $\mathbf{x}^{\mathbf{p}} = x_1^{p_1} \cdots x_n^{p_n}$;
the scalar product $\langle \mathbf{p}, \mathbf{q} \rangle = p_1 q_1 + \cdots + p_n q_n$;
$\|\mathbf{p}\| = |p_1| + \cdots + |p_n|$.

1. Resonant normal form (1)

Consider a Hamiltonian system

$$\dot{\xi}_j = \frac{\partial \gamma}{\partial \eta_j}\,, \quad \dot{\eta}_j = -\frac{\partial \gamma}{\partial \xi_j}\,, \quad j = 1, \ldots, n \tag{1}$$

with n degrees of freedom in the neighborhood of a stationary point at the origin

$$\boldsymbol{\zeta} \stackrel{\mathrm{def}}{=} (\boldsymbol{\xi}, \boldsymbol{\eta}) = 0. \tag{2}$$

If the Hamilton function $\gamma(\zeta)$ is analytic at this point, then it expands into a convergent power series

$$\gamma(\zeta) = \sum \gamma_{\mathbf{pq}} \boldsymbol{\xi}^{\mathbf{p}} \boldsymbol{\eta}^{\mathbf{q}}, \tag{3}$$

where $\mathbf{p}$, $\mathbf{q} \in \mathbb{Z}^n$, $\mathbf{p}, \mathbf{q} \geqslant 0$, $\gamma_{\mathbf{pq}}$ are constant coefficients. Since the point (2) is stationary, the expansion of (3) starts with quadratic terms. They correspond to the linear part of the system (1).

The eigenvalues of its matrix are divided into pairs $\lambda_{j+n} = -\lambda_j$, $j = 1, \dots, n$. Denote by vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ the set of ***basic eigenvalues***. As known, canonical coordinate substitutions $\boldsymbol{\xi}, \boldsymbol{\eta} \longrightarrow \mathbf{x}, \mathbf{y}$ preserve the Hamiltonian nature of the system.

Theorem 1 ([Bruno, 1972, §12]).

There is a canonical formal transformation $\boldsymbol{\xi}, \boldsymbol{\eta} \leftrightarrow \mathbf{x}, \mathbf{y}$ *that reduces the Hamiltonian* (3) *to the normal form*

$$g(\mathbf{x}, \mathbf{y}) = \sum g_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}}, \qquad (4)$$

where the series g *contains only resonant terms satisfying* ***resonant equation*** $\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0$.

For the real initial system (1), the constant coefficients $g_{\mathbf{pq}}$ of the complex normal form (4) satisfy special realness relations, and the standard canonical linear coordinate substitution $\mathbf{x}, \mathbf{y} \to \mathbf{X}, \mathbf{Y}$ reduces the system (4) into a real system.

Definition 1.

For each resonance are defined:

- ***multiplicity*** $\mathfrak{k}$: the number of linearly independent solutions $\mathbf{p} \in \mathbb{Z}^n$ to the resonant equation
$$\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0 \tag{5}$$
- ***order*** $\mathfrak{q}$: $\mathfrak{q} = \min |\mathbf{p}|$ over $\mathbf{p} \in \mathbb{Z}^n \backslash \{0\}$, satisfying (5);
- n-***frequency resonance***: if exactly n nonzero eigenvalues λ_j are included in the nontrivial solution of the resonance equation;
- ***strong resonances*** are called the resonances of orders 2, 3 or 4.

Condition A_k^n

for system with n DOF takes place if the resonant equation $\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0$ has no integer solutions $\mathbf{p} \in \mathbb{Z}^n$ with $\|\mathbf{p}\| \leq k$.

This condition means that there are no resonances up to and including the order k. If it is satisfied, then in the normal form (4)

$$g = \sum_{l=1}^{[k/2]} g_l(\boldsymbol{\rho}) + \tilde{g}^{(k)}(\mathbf{z}, \bar{\mathbf{z}}), \tag{6}$$

where $g_l(\boldsymbol{\rho})$ are homogeneous polynomials from $\rho_j = i z_j \bar{z}_j$, $j = 1, \ldots, n$, of degree l, and $\tilde{g}^{(k)}$ is a series from $\mathbf{z}, \bar{\mathbf{z}}$ starting with powers above k.

In particular, under the condition A_2^n we have

$$g = \langle \boldsymbol{\rho}, \boldsymbol{\lambda} \rangle + \tilde{g}^{(3)}(\mathbf{z}, \bar{\mathbf{z}}). \tag{7}$$

And under the condition A_4^n we have

$$g = \langle \boldsymbol{\rho}, \boldsymbol{\lambda} \rangle + \langle C\boldsymbol{\rho}, \boldsymbol{\rho} \rangle + \tilde{g}^{(5)}(\mathbf{z}, \bar{\mathbf{z}}), \tag{8}$$

where C is $n \times n$ matrix.

Definition 2.

A stationary point (SP) $\zeta = 0$ of a real Hamiltonian system (1) is ***stable by Lyapunov*** if for every $\varepsilon > 0$ in "cube" $\|\zeta\| < \varepsilon$ there exists a closed integral $(2n-1)$-dimensional manifold $\mathcal{L}$ surrounding the point $\zeta = 0$ from all sides.

Lemma 1.

A stationary point $\zeta = 0$ *is* ***Lyapunov stable*** *if there exists a sign-definite real integral*

$$f(\zeta) = f_l(\zeta) + \tilde{f}^{(l)}(\zeta) \tag{9}$$

of the system (1)*, where* $f_l(\zeta)$ *is a homogeneous form of degree* l*. In other words,*

$$\{f, \gamma\} = 0, \tag{10}$$

where $\{\cdot, \cdot\}$ *is the Poisson bracket, and* $f_l(\zeta)$ *does not equal to zero at any* ζ *except the point* $\zeta = 0$*.*

Stability is possible only if $\operatorname{Re}\boldsymbol{\lambda} = 0$. If the condition A_2^n is satisfied, then all λ_j are different and non-zero. In this case the complex coordinates $\mathbf{x}, \mathbf{y}$ are related to the real coordinates $\mathbf{X}, \mathbf{Y}$ by the canonical substitution

$$x_j = \frac{iX_j - Y_j}{\sqrt{2i}},\ y_j = \frac{iX_j + Y_j}{\sqrt{2i}},\ j = 1, \dots, n.$$

With complex conjugation

$$\bar{x}_j = -iy_j, \quad \bar{y}_j = -ix_j, \quad j = 1, \dots, n,$$

the Hamiltonian function $g(\mathbf{x}, \mathbf{y})$ goes into itself, i.e. into (4):

$$g_{\mathbf{pq}} = (-i)^{\|\mathbf{p}+\mathbf{q}\|} \bar{g}_{\mathbf{qp}},$$

as far as $p_j, q_j \geqslant 0$.

Suppose $X_j^2 + Y_j^2 = R_j$, $\lambda_j = i\alpha_j$, $j = 1, \ldots, n$. Then in real coordinates $R_j \geqslant 0$, α_j is real,

$$\begin{aligned} \rho_j =& x_j y_j = \frac{i}{2}\left(X_j^2 + Y_j^2\right) = \frac{i}{2} R_j, \quad j = 1, \ldots, n, \\ \langle \boldsymbol{\lambda}, \boldsymbol{\rho} \rangle =& \sum_{j=1}^{n} \lambda_j \rho_j = -\frac{1}{2} \sum_{j=1}^{n} \alpha_j \left(X_j^2 + Y_j^2\right) = -\frac{1}{2} \langle \boldsymbol{\alpha}, \mathbf{R} \rangle . \end{aligned} \tag{11}$$

Theorem 2 ([Lejeune Dirichlet, 1846]).

If the condition A_2^n is satisfied and the numbers $\alpha_1, \ldots, \alpha_n$ are of the same sign, then the stationary point $\boldsymbol{\zeta} = 0$ is stable according to Lyapunov.

Here the role of the integral f is played by the Hamiltonian γ itself, for it is an integral, the notation (7) has the form (6) with $k = 2$ and the form

$$\gamma_2 = g_2 = -\frac{1}{2}\sum_{j=1}^{n} \alpha_j R_j = -\frac{1}{2}\langle \boldsymbol{\alpha}, \mathbf{R} \rangle$$

is sign-defined, for $\mathbf{R} \geqslant 0$.

By ***formal*** we mean power series, about the convergence of which nothing is known.

Definition 3 ([Moser, 1958]).

A stationary point (2) of a real Hamiltonian system (1) is ***formally stable*** if there exists a formal real sign-defined integral (9) of the system (1), i.e., the formal identity (10) is satisfied and the homogeneous form f_l is null only at $\zeta = 0$.

Formal stability means that the scattering of solutions from the stationary point is very slow: slower than any finite degree of t.

Definition 4 ([Markeev, 1978, Ch. 4, § 4]).

A stationary point (2) of a real Hamiltonian system (1) is ***formally stable*** if there exists a formal real integral

$$f(\zeta) = f_l(\zeta) + f_{l+1}(\zeta) + \ldots + f_m(\zeta) + \tilde{f}^{(m)}(\zeta)$$

of system (1), where $f_k(\zeta)$ are homogeneous forms of degree k and the sum

$$f^*(\zeta) = f_l + f_{l+1} + \ldots + f_m \tag{12}$$

does not equal to zero in some neighborhood of the point $\zeta = 0$ besides it.

Definition 5 ([Bruno, Batkhin, 2012]).

A point ζ^0 is called a ***root of order*** k of a polynomial $\hat{f}(\zeta)$ if at this point the $\hat{f}$ itself and all its partial derivatives up to and including order k are zero, but at least one derivative of order $k+1$ is nonzero.

Conjecture 1.

If a polynomial (12) *with* $m > l$ *does not converge to zero in some neighborhood of point* $\zeta = 0$ *except it, then every root* (ζ) *of the polynomial* f_l *other than* $\zeta = 0$ *has an even multiple.*

Since $\rho_j \rho_k = -\dfrac{1}{4} R_j R_k$, then under the condition A_4^n the sum (8) takes the form

$$g = -\frac{1}{2} \langle \boldsymbol{\alpha}, \mathbf{R} \rangle - \frac{1}{4} \langle C\mathbf{R}, \mathbf{R} \rangle + \tilde{g}^{(5)}. \qquad (13)$$

Hence, all elements of matrix C are real.

Let $K \subset \mathbb{R}^n$ be a linear shell of integers $\mathbf{q}$ satisfying the equation $\langle \boldsymbol{\alpha}, \mathbf{q} \rangle = 0$, and $Q = \{\mathbf{q} \geqslant 0,\ \mathbf{q} \neq 0\} \subset \mathbb{R}^n$ is a non-negative orthant without origin.

Theorem 3 ([Bruno, 1967]).

If Condition A_4^n *is satisfied and in* (13)

$$\langle C\mathbf{q}, \mathbf{q}\rangle \neq 0 \textit{ for } \mathbf{q} \in K \cap Q,$$

then the point $\boldsymbol{\zeta} = 0$ *is formally stable in the sense of Definition 3*

Here, the normal form of the Hamiltonian (4) from Theorem 1 is used to construct the formal integral.

According to (11) in real coordinates, the normal form (6) is

$$g = -\frac{1}{2}\langle \boldsymbol{\alpha}, \mathbf{R}\rangle + \sum_{l=2}^{[k/2]} h_l(\mathbf{R}) + \tilde{g}^{(k)}, \tag{14}$$

where the homogeneous polynomials $h_l = (i/2)^l g_l(\mathbf{R})$ are real. The following generalization of Theorem 3 is proved verbatim like it.

Theorem 4.
If the condition A_k^n is satisfied and in the normal form (14)

$$\sum_{l=2}^{[k/2]} h_l(\mathbf{R}) \neq 0 \text{ for } \mathbf{R} \in K \cap Q,$$

then the point $\boldsymbol{\zeta} = 0$ is formally stable in the sense of Definition 4.

3. Formal stability. Investigation in a generic case with 3DOF (1)

Consider a Hamiltonian system in the vicinity of the SP for which the following conditions are satisfied:

- the number of degrees of freedom of the system is greater than two;
- the quadratic form γ_2 in expansion (3) is nondegenerate and is not definite;
- the Hamiltonian function γ smoothly depends of the vector of parameters $\mathbf{P}$ from a domain $\Pi \subset \mathbb{R}^m$.

3. Formal stability. Investigation in a generic case with 3DOF (2)

Corollary 1 (of Formal Stability Theorem 3).

If under the condition of Theorem 3 in $\mathbb{R}^3$ *the intersection of the plane* $\langle \boldsymbol{\lambda}, \mathbf{q} \rangle = 0$ *and the cone* $\langle C\mathbf{q}, \mathbf{q} \rangle$ *either does not belong to* $\mathcal{Q}$*, or belongs to* $\mathcal{Q} = \mathbb{R}^3_+$*, but does not contain the integral vector* $\mathbf{q}$*, then the SP is formally stable.*

Definition 6.

A ***resonant variety*** $\mathcal{R}_n^{\mathbf{p}}$ in the space K of coefficients $a_1, \dots, a_n$ of the semi-characteristic polynomial $\chi_n(\mu)$ of degree n is an algebraic variety, on which the vector of basis eigenvalues $\boldsymbol{\lambda}$ is a nontrivial solution to the resonant equation $\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0$ for a fixed integer vector $\mathbf{p}^* \in \mathbb{Z}^n \backslash \{0\}$. An analytical representation of the variety $\mathcal{R}_n^{\mathbf{p}*}$ in an implicit or parametric form is denoted by $R_n^{\mathbf{p}^*}$.

3. Formal stability. Investigation in a generic case with 3DOF (3)

To examine the formal stability of a SP of a Hamiltonian system (1), we should do the following steps [Batkhin, Khaydarov, 2023]:

- find in the space of parameters Π the stability set Σ of the linear system;
- find such domains, in which the quadratic form $\gamma_2(\mathbf{z})$ is not sign definite;
- find parts S_k in these domains that do not contain strong resonances;
- normalize the Hamiltonian in each of these parts S_k up to order four, and
- apply *Formal Stability Theorem* 3.

To do this, it is sufficient to select a point in each S_k in the space of parameters and use one of the normalization algorithms for the Hamiltonian function. Since all eigenvalues λ_k, $k = 1, \ldots, n$, are simple at each interior point of S_k, the invariant normalization algorithm can be easily applied.

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Let the function $f(t)$ be defined at real $t \to -\infty$. It is said to have ***order*** $\delta = \delta(t)$ if $\delta = \inf \varepsilon$ such that $f(t)/(-t)^{\varepsilon} \to 0$ at $t \to -\infty$. If $\delta > 0$, then $f(t)$ is unbounded, if $\delta < 0$, then $f(t) \to 0$ at $t \to -\infty$. In the latter case $\delta(f) < 0$, the larger δ is, the slower $f(t)$ approaches zero.

Definition 7.

Let the solution $\zeta(t)$ of the Hamiltonian system (1) tends to a SP (2) at $t \to -\infty$. On this solution ***order of scattering*** $\Delta = \min \{\delta \|\zeta\|\}$.

Definition 8.

The scattering order $\widetilde{\Delta}$ ***of solutions*** of the system (1) from the SP (2) is the lower bound of the scatter order Δ over all solutions $\zeta(t)$ that tend to the point (2) at $t \to -\infty$.

The smaller $\widetilde{\Delta} < 0$, the faster the solutions are scattered from the SP. At formal stability the order of scattering of solutions from the SP is zero. Let us estimate the order of scattering $\widetilde{\Delta}$ in the absence of formal stability. The cases $-10^{-10} < \widetilde{\Delta} < 0$ can be considered as ***weak stable***.

Conjecture 2.

Let the condition A_2^n *and* $\varkappa = \min \|\mathbf{p} + \mathbf{q}\| > 2$ *by integer solutions* $\mathbf{p} \geqslant 0$, $\mathbf{q} \geqslant 0$ *of equation* $\langle \boldsymbol{\alpha}, \mathbf{p} - \mathbf{q} \rangle = 0$ *be satisfied, then the order of scatter of the system solutions* (1) *from the SP* $\widetilde{\Delta} \geqslant (2 - \varkappa)^{-1}$.

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together with:

1. more details, with examples;
2. number-theoretical approach simplifying the proofs of formal stability;
3. computing of formal stability in a complicated case;
4. similar theory for a neighborhood of a periodic solution.

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